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Utility Duality under Additional Information: Conditional Measures versus Filtration Enlargements

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Abstract

The utility maximisation problem is considered for investors with anticipative additional information. We distinguish between models with conditional measures and models with enlarged filtrations. The dual functions of the maximal expected utility are determined with the help of f-divergences. We assume that our measures are absolutely continuous with respect to a local martingale measure (LMM), but not necessarily equivalent. Thus we do *not* exclude arbitrage.

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1 Introduction

In this paper we consider the utility maximisation problem for investors on financial markets who have some anticipative knowledge. These are investors who have some information which is relevant for the future price development, but which is not official and can not be obtained by observing

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the price process or by reading the newspaper. Think, for example, of an investor knowing that a big trader is trying to keep the price of a certain asset below a fixed level during the next two weeks. More generally, suppose an investor knows that an event A will occur. In this case, he perceives the price process under a conditional measure $P(\cdot|A)$, where P is the original measure.

A different situation is given if an investor thinks about consulting an expert who has some knowlegde being of relevance for the financial market. For simplicity, suppose that the expert knows whether the event A will happen or not, and assume that this information would cost $p > 0$. In order to decide whether the investor should buy the information, we have to determine his additional expected utility *before* he gets the expert's information. This can be done by enlarging the set of his strategies: We allow him not only to use $(\mathcal{F}_t)$ adapted strategies, but also strategies which are adapted to the *enlarged* filtration $(\mathcal{F}_t \vee \{A, A^c\})$. Summarising, we can distinguish two cases:

1. The investor knows that the event A will *certainly* happen. In this case he perceives the price process from the conditional measure $P(\cdot|A)$.

2. The investor *will* know whether A will happen or not. His information flow is represented by the enlarged filtration $(\mathcal{F}_t \vee \sigma\{A, A^c\})$.

The first case describes the situation *after* the investor got the extra information, and the second case *before* he gets the additional information. In the second case the measure will switch either to $P(\cdot|A)$ or $P(\cdot|A^c)$. Let $u_A(x)$ and $u_{A^c}(x)$ be the maximal expected utility under P conditioned on A and A^c respectively. By taking the average we obtain the expected utility in the second case, namely $P(A)u_A(x) + P(A^c)u_{A^c}(x)$. Thus *initial* enlargements of filtrations can be reduced to maximising the utility relative to conditional measures.

In order to solve the classical utility maximisation problem researchers have applied convex duality methods. We choose as a starting point the work by Kramkov and Schachermayer, [15] and [16], where the duality is anlaysed in the framework of the semimartingale model. Moreover, they represent the convex conjugate function of the maximal expected utility with the help of the equivalent local martingale measures (ELMM) of the underlying asset prices.

What about the utility maximisation problem under additional information? There exist solutions for enlarged filtrations. The first are [8] and [18]. Since then, their work was generalised by many authors. Just to mention a few, see [2], [1], [13] [12]. Most of these papers consider initial enlargements. Non-initial enlargements have been considered recently in [7], [5] and [4].

Baudoin introduced the concept of *weak information* (see [6]). In his model, the investors know in advance the *distribution* μ of a random variable G. The maximal expected utility has then to be maximised with respect to the whole *set* of equivalent measures under which the conditional distribution of G is given by μ. As a consequence, weak information leads to *market uncertainty*. In our case, however, we have certainty about the measure.

So far, most approaches assume that under the new information the market is still free of arbitrage, and hence, that there exist ELMMs. Thus, the maximisation problem can be solved with the classical methods. In this paper we will only assume that the conditional measures are *absolutely continuous* relative to P, i.e. $P(\cdot|A) \ll P$. As a consequence, ELMMs do *not* exist. However, the conditional measures are absolutely continuous with respect to a LMM. This is, as we will see, sufficient for deriving formulas of the dual function similar to the classical representations.

2 The model: starting from complete markets

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space, where $(\mathcal{F}_t)$ is a filtration satisfying the usual conditions. Let S be a continuous price process starting in zero and being a $(\mathcal{F}_t)$-semimartingale with decomposition $S = M + \alpha \cdot \langle M, M \rangle$. We assume the market to be complete. I.e. there exists a unique measure Q such that S is a local martingale relative to $(\mathcal{F}_t)$ and Q.

We denote by $\mathcal{A}(\mathcal{F})$ the set of all $(\mathcal{F}_t)$-predictable processes θ which satisfy $\theta_0 = 0$ and which are integrable with respect to S and $(\mathcal{F}_t)$ in the usual sense (see Protter [19]). The elements of $\mathcal{A}(\mathcal{F})$ will be called *strategies*. Moreover, a strategy is called *a-admissible* if the stochastic integral process satisfies $(\theta \cdot S)_t \geq -a$, for all $t \geq 0$. We fix a time horizon $T > 0$ and aim at maximising the utility of the wealth at time T. Given a utility function U, the maximal expected utility is defined by

$$u^{\mathcal{F}}(x) = \sup\{EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{A}(\mathcal{F}) \text{ is } x - \text{admissible}\}.$$

We restrict the class of utility functions in order to simplify our analysis: let U be strictly increasing, strictly concave and continuously differentiable on $(0, \infty)$. Furthermore we assume that U satisfies the Inada conditions

$$\lim_{x \downarrow 0} U'(x) = \infty \qquad \text{and} \qquad \lim_{x \to \infty} U'(x) = 0. \tag{1}$$

Moreover let

$$V(y) = \sup_{x > 0} \left[U(x) - xy \right], \quad y \in \mathbb{R},$$

be the convex conjugate function of $-U(-x)$. Due to (1), V is the *Legendre conjugate*, i.e. $V(y) = U((U')^{-1}(y)) + y\ (U')^{-1}(y)$, $y > 0$. It can be shown that V is a continuously differentiable, decreasing and strictly convex function satisfying $\lim_{y\downarrow 0} V'(y) = -\infty$ and $\lim_{y\to\infty} V'(y) = 0$. Furthermore, U is the Legendre conjugate of V, i.e. $U(x) = V((V')^{-1}(-x)) + x\ (V')^{-1}(-x)$, $x > 0$. In the sequel, any pair of functions satisfying the mentioned properties of (U, V) will be called *Legendre pair.*

Note that also the function $u^{\mathcal{F}}(x)$ is concave on $(0, \infty)$. We can therefore again define the conjugate

$$v^{\mathcal{F}}(y) = \sup_{x>0} \left[u^{\mathcal{F}}(x) - xy \right], \quad y \in \mathbb{R}.$$

A sufficient and necessary condition for $(u^{\mathcal{F}}, v^{\mathcal{F}})$ to be a Legendre pair was given by Kramkov, Schachermayer in [16].

Theorem 2.1. *(Theorem 1 and 2 in [16])*
$(u^{\mathcal{F}}, v^{\mathcal{F}})$ is a Legendre pair if and only if $v^{\mathcal{F}}(y) < \infty$ for all $y > 0$. Moreover,

$$v^{\mathcal{F}}(y) = EV\left(y\frac{dQ}{dP} \right).$$

This theorem implies that the maximal expected utility depends on how close the real measure P is to the risk neutral measure Q. Indeed, the value $EV\left(y\frac{dQ}{dP}\right)$ is a well-known distance between the two probability measures Q and P, namely a so-called divergence. We now collect some basic definitions we will need in the following.

Definition 2.2. Let μ and ν be probability measures on the measurable space $(\Omega, \mathcal{F})$. Moreover, let $f : (0, \infty) \to \mathbb{R}$ be a convex function, $f(0) = \lim_{x\downarrow 0} f(x)$, and $\mathcal{A}$ a sub-σ-algebra of $\mathcal{F}$. The *f-divergence* of μ relative to ν on $\mathcal{A}$ is defined as

$$f_{\mathcal{A}}(\mu\|\nu) = \begin{cases} \int f\left(\frac{d\mu}{d\nu}\Big|_{\mathcal{A}} \right) d\nu, & \text{if } \mu \ll \nu \text{ on } \mathcal{A} \text{ and the integral exists,} \\ \infty, & \text{else.} \end{cases}$$

If $f(x) = x \log x$, then $f(\mu\|\nu)$ coincides with the entropy of μ relative to ν. Crucial for us will be the next definition.

Definition 2.3. The *reverse function* of the convex function f is defined by

$$\hat{f}(x) = xf\left(\frac{1}{x}\right), \quad x \in (0, \infty).$$

Again we set $\hat{f}(0) = \lim_{x\downarrow\infty} \hat{f}(x)$.

Lemma 2.4. *If f is strictly convex and differentiable on $(0, \infty)$, then also the reverse function $\hat{f}$ is strictly convex and differentiable on $(0, \infty)$. Moreover, if $P \sim Q$, then*

$$f_{\mathcal{A}}(P\|Q) = \hat{f}_{\mathcal{A}}(Q\|P).$$

Proof. For a proof of these properties see Lemma 1 in [10], or Theorem 1.13 in [17]. □

The conjugate function $v^{\mathcal{F}}(y)$ in Thereom 2.1 is given as divergence with respect to the convex function $x \mapsto V(yx)$. For simplicity we will use the notation $V_y(x) = V(yx)$, $x \geq 0$. Thus

$$v^{\mathcal{F}}(y) = (V_y)_{\mathcal{F}_T}(Q\|P) = (V_y)\hat{}_{\mathcal{F}_T}(P\|Q). \tag{2}$$

Finally, observe that the function $V_y(x)$ is also convex in y, i.e. for all $\lambda \in (0,1)$, and $x, y, z > 0$ we have

$$V_{\lambda y + (1-\lambda)z}(x) \leq \lambda V_y(x) + (1-\lambda)V_z(x).$$

Conditional measures

Now let us consider the utility maximisation problem under conditional measures. For this let $A \in \mathcal{F}$ and $P_A = P(\cdot|A) = \frac{1}{P(A)}P(\cdot \cap A)$. If the conditional measure P_A is equivalent to P, then we obtain the dual function by replacing P with P_A in equation (2). We will prove that if they are not equivalent, then the dual function still can be written as the reverse V_y-divergence of P_A relative to Q.

Theorem 2.5. *Let $v_A(y) = (V_y)\hat{}_{\mathcal{F}_T}(P_A\|Q)$, $y > 0$. If $v_A(y) < \infty$ for all $y > 0$, then (u_A, v_A) is a Legendre pair.*

Enlargement of filtrations

We will derive a similar result for an investor with information given by the filtration

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(G),\ t \geq 0,$$

where G is an arbitrary random variable with values in a polish space Γ. For this let π be a regular conditional probability of P with respect to $\sigma(G)$. Hence $\pi(\cdot, \omega)$ is a probability measure and $\omega \mapsto \pi(A, \omega)$ is $\sigma(G)$-measurable for all $A \in \mathcal{F}$.

If $(V_y)\hat{}\,(\pi(\cdot,\omega)\|Q) < \infty$, $y > 0$, then this function is the conjugate of the maximal expected utility $u_{\pi(\omega)}$ relative to $\pi(\cdot, \omega)$. We will show that there exists a $\sigma(G)$-measurable process Z such that $Z(x) = u_{\pi(\omega)}(x)$, a.s. This allows us to derive a *stochastic* dual representation.

Theorem 2.6. *Let* $Y(y) = (V_y)\hat{}_{\mathcal{F}_T}(\pi(\cdot,\omega)\|Q)$, $y > 0$. *If* $Y(y) < \infty$, *a.s, then* (Z, Y) *is a Legendre pair, a.s.*

Note that $Y(y) < \infty$, a.s, implies that $\pi(\cdot,\omega) \ll P$, for almost all ω. This is similar to a condition introduced by Jacod in [14]. It guarantees that S remains a semimartingale with respect to $(\mathcal{G}_t)$.

3 Restricting complete markets

Let $A \in \mathcal{F}$ and denote by $P_A = P(\cdot|A) = \frac{1}{P(A)}P(\cdot \cap A)$ the conditional measure on A. We suppose throughout this section that P_A is absolutely continuous with respect to P on $\mathcal{F}_T$, i.e. $P_A|_{\mathcal{F}_T} \ll P|_{\mathcal{F}_T}$. Observe that we do *not* assume that A belongs to $\mathcal{F}_1$.

Let $Z_t = \frac{1}{P(A)}P(A|\mathcal{F}_t)$, $t \in [0,1]$. The Girsanov-Lenglart Theorem implies that

$$N_t = M_t - \int_0^t \frac{1}{Z_u} d\langle M, Z\rangle_u$$

is a local martingale relative to P_A. Note that $\langle M, M\rangle = \langle N, N\rangle$, P_A-a.s. Moreover, there exists an $(\mathcal{F}_t)$-predictable process β such that $\beta \cdot \langle M, M\rangle = \frac{1}{Z} \cdot \langle M, Z\rangle$. Therefore, the dynamcis of the price S under the measure P_A is given by

$$S_t = S_0 + N_t + \int_0^t (\alpha_s + \beta_s) d\langle N, N\rangle_s.$$

Surprisingly, if $A \in \mathcal{F}_T$ and $P(A) < 1$, then there exists no LMM of S which is equivalent to P_A. Suppose that there exists a LMM R which is equivalent to P_A. Then the convex combination $R^\mu = \mu R + (1-\mu)Q$ is a LMM equivalent to P. Moreover, since $P(A) < 1$, we have $R^\mu \neq Q$. This, however, is a contradiction to the completeness under P. One can even show that there exists arbitrage under P_A:

Theorem 3.1. *If* A *is* $\mathcal{F}_T$*-measurable and* $P(A) < 1$, *then the market admits arbitrage under* P_A.

Proof. Let $c > 0$, and $f = 1_A - c1_{A^c}$ such that $E^P[f] = 0$. Due to completeness, there exists a strategy θ such that $(\theta \cdot S)_T = f$. Moreover, $(\theta \cdot S)_t = E^Q[f|\mathcal{F}_t] \geq -c$, which shows that θ is admissible. Note that under P_A, $(\theta \cdot S)_T = 1$. Therefore θ is an arbitrage strategy relative to P_A. □

We now aim at determining the maximal expected utility under the measure P_A. For this let $(\mathcal{F}_t^A)$ the smallest filtration containing $(\mathcal{F}_t \vee \sigma\{A, A^c\})$

and all P_A-null sets, and put

$$u_A(x) \quad = \quad \sup\{E^{P_A}[(U(x + (\theta \cdot S)_T)] : \theta \ (\mathcal{F}_t^A)\text{-predictable}, \\ x\text{-adm. and } S\text{-integrable relative to } P_A\}.$$

We start with the following observation.

Lemma 3.2. *$u_A(x)$ is equal to the maximal expected utility where the supremum is taken only over all $(\mathcal{F}_t)$-predictable processes which are x-admissible and integrable with respect to P. As a consequence, $u_A(x) < \infty$ for all $x > 0$.*

Proof. We have to show that $u_A(x)$ is equal to

$$\tilde{u}(x) \quad = \quad \sup\{E^{P_A}[(U(x + (\theta \cdot S)_T)] : \theta \ (\mathcal{F}_t)\text{-predictable}, \\ x\text{-adm. and } S\text{-integrable relative to } P\}.$$

Every strategy integrable relative to P is also integrable relative to P_A (see f.e. Theorem 25, Chapter IV in [19]), and therefore $\tilde{u}(x) \leq u_A(x)$. For the reverse inequality let $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\{A, A^c\}$. Denote by $\mathcal{P}(\mathcal{F})$ and $\mathcal{P}(\mathcal{G})$ the predictable σ-fields with respect to $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ respectively. We show at first

$$\mathcal{P}(\mathcal{G}) = \{(\Delta \cap (A \times \mathbb{R}_+)) \cup (\Gamma \cap (A^c \times \mathbb{R}_+)) : \Delta,\ \Gamma \in \mathcal{P}(\mathcal{F})\}. \qquad (3)$$

Note that the RHS is a σ-algebra which is contained in $\mathcal{P}(\mathcal{G})$. Moreover, each set of the form $(B \cap A)\times]s, t]$ or $(B \cap A^c)\times]s, t]$, $B \in \mathcal{F}_s$, belongs to the RHS. Therefore, $\mathcal{P}(\mathcal{G})$ is a subset of the RHS, and hence equation (3) holds.

A monotone class argument implies that every $(\mathcal{G}_t)$-predictable process may be written as a sum of the form $1_A\zeta + 1_{A^c}\eta$, where ζ and η are $(\mathcal{F}_t)$-predictable. Now assume that θ is an x-admissible and bounded strategy. It is straightforward to show, via stopping for example, that ζ and η may be chosen to be bounded, x-admissible and hence S-integrable relative to P.

As a consequence, the maximal expected utility obtained by using $(\mathcal{G}_t)$-predictable strategies is equal to $\tilde{u}(x)$ (here we also use the fact that $\tilde{u}(x)$ can be attained with bounded strategies). Since $(\mathcal{F}_t^A)$ is the filtration $(\mathcal{G}_t)$ completed by the P_A-null sets, the proof is complete. □

We define now

$$v_A(y) = (V_y)\hat{}\,(P_A \| Q),$$

and for the rest of the section we assume that $v_A(y) < \infty$ for all $y > 0$. We have to show that (u_A, v_A) is a Legendre pair. We will deduce this from

Thereom 2.1. For this we approximate the measure P_A with the measures $P_\varepsilon = (1-\varepsilon)P_A + \varepsilon P$, $\varepsilon \in (0,1)$. Observe that P_ε is equivalent to P. Moreover, the measure Q is the unique ELMM of S relative to P_ε. Therefore, we may apply the Thereom 2.1 to the maximal expected utility under P_ε. By taking limits we will obtain the result.

At first we define

$$\begin{aligned} u_\varepsilon(x) &= \sup\{E^{P_\varepsilon}[(U(x+(\theta\cdot S)_T)] : \theta\ (\mathcal{F}_t)\text{-predictable}, \\ &\qquad x\text{-adm. and } S\text{-integrable relative to } P\} \end{aligned}$$

and show that $u_\varepsilon(x)$ converges to $u_A(x)$.

Lemma 3.3. *Let $x > 0$. Then*

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon(x) = u_A(x).$$

Proof. Let θ be $(\mathcal{F}_t)$-predictable, S-integrable and x-admissible relative to P. Put $X_T = x + (\theta\cdot S)_T$ and assume that $U(X_T)$ is P-integrable. Then on the one hand, $\lim_{\varepsilon\downarrow 0} E^{P_\varepsilon}[U(X_T)] = E^{P_A}[U(X_T)]$, and therefore $\liminf_{\varepsilon\downarrow 0} u_\varepsilon(x) \geq u_A(x)$ (see Lemma 3.2).

On the other hand, $E^{P_\varepsilon}[U(X_T)] = \varepsilon E^P[U(X_T)] + (1-\varepsilon)E^{P_A}[U(X_T)]$, which implies

$$u_\varepsilon(x) \leq \varepsilon\, u(x) + (1-\varepsilon)u_A(x).$$

Consequently, $\limsup_{\varepsilon\downarrow 0} u_\varepsilon(x) \leq u_A(x)$, and thus the result. □

Now let $v_\varepsilon(y)$ be the conjugate function of $u_\varepsilon(x)$. By Theorem 2.1 we have

$$v_\varepsilon(y) = E^P V\left(y\frac{dP_\varepsilon}{dQ}\right) = (V_y)\hat{}_{\mathcal{F}_T}\ (P_\varepsilon\|Q)$$

We will show that v_ε converges to v_A as $\varepsilon \downarrow 0$. But before, we need the following.

Lemma 3.4. *The function v_A is decreasing, differentiable and strictly convex on $(0,\infty)$. Moreover,*

$$\lim_{y\downarrow 0} v_A'(y) = -\infty, \quad \lim_{y\to\infty} v_A'(y) = 0.$$

Proof. Our assumptions on U imply that V, and thus $y \mapsto (V_y)\hat{}(z)$ (with $z > 0$), is strictly convex, decreasing and differentiable on $(0,\infty)$. Therefore, also $v_A(y)$ is strictly convex and decreasing on $(0,\infty)$.

It is known that for any convex function the left and right derivatives exist on the interior of the domain. Thus $v_A(y)$ has a right derivative $\frac{d}{dy+}v_A(y)$

and a left derivative $\frac{d}{dy-}v_A(y)$ on $(0,\infty)$. Note that $\frac{1}{h}[(V_y)\hat{} - (V_{y+h})\hat{}]$ is increasing to $-\frac{d}{dy}(V_y)\hat{}$ as $h \downarrow 0$. As a consequence, $\frac{d}{dy}(V_y)\hat{}\left(\frac{dP_A}{dQ}\right)$ is in $L^1(Q)$. Moreover

$$\begin{aligned}\frac{d}{dy+}v_A(y) &= \int \frac{d}{dy+}(V_y)\hat{}\left(\frac{dP_A}{dQ}\right)\, dQ = \int \frac{d}{dy-}(V_y)\hat{}\left(\frac{dP_A}{dQ}\right)\, dQ \\ &= \frac{d}{dy-}v_A(y),\end{aligned}$$

showing that v_A is differentiable on $(0,\infty)$. Since $\lim_{y\downarrow 0} V'(y) = -\infty$, monotone convergence implies $\lim_{y\downarrow 0} v_A'(y) = \lim_{y\downarrow 0}\int \frac{d}{dy}(V_y)\hat{}\left(\frac{dP_A}{dQ}\right)\, dQ = -\infty$. Finally, since $\lim_{y\to\infty} V'(y) = 0$, we obtain with dominated convergence, $\lim_{y\to\infty} v_A'(y) = \lim_{y\to\infty}\int \frac{d}{dy}(V_y)\hat{}\left(\frac{dP_A}{dQ}\right)\, dQ = 0$. □

We now prove the remaining properties needed for (u_A, v_A) to be a Legendre pair.

Lemma 3.5. *The function v_A is the dual function of u_A, i.e.*

$$\begin{aligned}v_A(y) &= \sup_{x>0}\left[u_A(x) - xy\right], \quad y>0,\\ u_A(x) &= \inf_{y>0}\left[v_A(v) + xy\right], \quad x>0.\end{aligned}$$

Moreover, u_A is strictly concave and differentiable on $(0,\infty)$, and

$$\lim_{x\downarrow 0} u_A'(x) = \infty, \qquad \lim_{x\to\infty} u_A'(x) = 0.$$

As a consequence, (u_A, v_A) is a Legendre pair; and Theorem 2.5 is shown.

Proof. By Lemma A.2 there exists a non-negative and convex function g such that all g-divergences coincide with the $(V_y)\hat{}$-divergences up to some constant. To simplify notation, we assume that $(V_y)\hat{}$ is already non-negative. Therefore, since $z \mapsto (V_y)\hat{}(z)$ is convex $(y>0)$,

$$\begin{aligned}(V_y)\hat{}\left(\frac{dP_\varepsilon}{dQ}\right) &\le (1-\varepsilon)(V_y)\hat{}\left(\frac{dP_A}{dQ}\right) + \varepsilon(V_y)\hat{}\left(\frac{dP}{dQ}\right)\\ &\le (V_y)\hat{}\left(\frac{dP_A}{dQ}\right) + (V_y)\hat{}\left(\frac{dP}{dQ}\right).\end{aligned}$$

Now dominated convergence implies for all $y>0$,

$$\lim_{\varepsilon\downarrow 0} v_\varepsilon(y) = \lim_{\varepsilon\downarrow 0}\int (V_y)\hat{}_{\mathcal{A}}\left(\frac{dP_\varepsilon}{dQ}\right)\, dQ = (V_y)\hat{}_{\mathcal{A}}\left(P_A \| Q\right) = v_A(y).$$

It remains to show that the limit v_A is indeed the conjugate function of u_A. By Theorem 2.1, the functions v_ε are differentiable and strictly convex on

$(0,\infty)$. Let w_ε and w denote the inverse function of the derivative of v_ε and v_A respectively. Convexity implies that on $(0,\infty)$ the derivatives of v_ε converge pointwise to the derivative of v_A (see Theorem 25.7 in [20]), and hence

$$\lim_{\varepsilon\downarrow 0} w_\varepsilon(z) = w(z).$$

Since $(u_\varepsilon, v_\varepsilon)$ is a Legendre pair,

$$u_\varepsilon(x) = v_\varepsilon(w_\varepsilon(-x)) + x w_\varepsilon(-x).$$

Moreover, v_ε converges uniformly on each closed bounded set of $(0,\infty)$ (see Theorem 10.8 in [20]). By letting $\varepsilon \downarrow 0$, we obtain with Lemma 3.3

$$u_A(x) = v_A(w(-x)) + xw(-x), \quad x > 0.$$

This shows that u_A is the dual function of v_A, and hence, that v_A is dual to u_A.

Since v_A is strictly convex, we have $\lim_{x\downarrow 0} u'_A(x) = \infty$ (see Theorem 26.3 in [20]). Finally, $\lim_{x\to\infty} u'_A(x) = 0$. If this were not the case, then $v_A(y) = \sup_{x>0}[u_A(x) - xy]$ were not finite for all $y > 0$, which is a contradiction to our assumption. Thus the proof is complete. □

We close this section by considering special utility functions.

Proposition 3.6. *(Logarithmic utility)*
Let $U(x) = \log x$ for all $x > 0$. Then $w(-x) = \frac{1}{x}$ and the maximal expected utility is equal to the relative entropy of P_A with respect to Q, i.e.

$$u_A(x) = \log(x) + \mathcal{H}(P_A\|Q).$$

Proof. Observe that the dual of the logarithm is given by $V(y) = -\log(y) - 1$, hence

$$(V_y)\hat{}(x) = x(-\log(\frac{y}{x}) - 1).$$

The conjugate is given by

$$\begin{aligned} v_A(y) &= -\log(y)\int \frac{dP_A}{dQ}\,dQ + \int \frac{dP_A}{dQ}\left(-1 + \log\frac{dP_A}{dQ}\right)dQ \\ &= -\log(y) - 1 + \int \log\frac{dP_A}{dQ}\,dQ \end{aligned}$$

and therefore $w(z) = -\frac{1}{z}$. Moreover, Theorem 2.5 implies

$$\begin{aligned} u_A(x) &= xw(-x) - 1 - \log(w(-x)) + \int \frac{dP_A}{dQ}\log\left(\frac{dP_A}{dQ}\right)dQ \\ &= \log(x) + \mathcal{H}(P_A\|Q). \end{aligned}$$

□

Proposition 3.7. *(Power utility)*
Let $0 < p < 1$ *and* $U(x) = \frac{1}{p}x^p$ *for all* $x \geq 0$. *Then*

$$u_A(x) = U(x) \left(\int \left(\frac{dP_A}{dQ} \right)^{\frac{1}{1-p}} dQ \right)^{1-p}.$$

Proof. Observe that $f(z) = (U')^{-1}(z) = (z)^{\frac{1}{p-1}}$ and

$$V(y) = U(f(y)) - y\,f(y) = \frac{1-p}{p}\, y^{-\frac{p}{1-p}},$$

and consequently

$$(V_y)\hat{}(x) \quad = \quad \frac{1-p}{p}\, x\, \left(\frac{y}{x}\right)^{-\frac{p}{1-p}} = \frac{1-p}{p}\, y^{-\frac{p}{1-p}}\, x^{\frac{1}{1-p}}.$$

Hence, the conjugate v_A satisfies

$$v_A(y) = \frac{1-p}{p}\, y^{-\frac{p}{1-p}} \int \left(\frac{dP_A}{dQ} \right)^{\frac{1}{1-p}} dQ.$$

Moreover, $w(z) = (v_A')^{-1}(z) = (-z)^{-(1-p)} \left(\int \left(\frac{dP_A}{dQ} \right)^{\frac{1}{1-p}} dR \right)^{(1-p)}, \quad z < 0.$
Therefore, by Theorem 2.5,

$$\begin{aligned} u_A(x) &= E\left[x w(-x) + v_A(w(-x)) \right] \\ &= x^p \left(\int \left(\frac{dP_A}{dQ} \right)^{\frac{1}{1-p}} dQ \right)^{1-p} \\ &\quad + \frac{1-p}{p}\, x^p \left(\int \left(\frac{dP_A}{dQ} \right)^{\frac{1}{1-p}} dQ \right)^{1-p} \\ &= U(x) \left(\int \left(\frac{dP_A}{dQ} \right)^{\frac{1}{1-p}} dQ \right)^{1-p}. \end{aligned}$$

□

Example 3.8. Let $(\Omega, \mathcal{F}, P)$ be the 1-dimensional canonical Wiener space equipped with the Wiener process $(W_t)_{0 \leq t \leq 1}$. More precisely, $\Omega = \mathcal{C}([0,1], \mathbb{R})$ is the set of continuous functions on $[0,1]$ starting in 0, $\mathcal{F}$ the σ-algebra of Borel sets with respect to uniform convergence, P the Wiener measure and W the coordinate process. Let $(\mathcal{F}_t)_{0 \leq t \leq 1}$ be the completed natural filtration generated by W.

We consider a financial asset with price process described by the stochastic exponential of W, i.e.

$$S_t = \mathcal{E}(W)_t, \quad 0 \le t \le 1.$$

Note that S satisfies (PRP) relative to $(\mathcal{F}_t)$. Moreover, let $A \in \mathcal{F}_1$ such that $P(A) > 0$. Then $\mathcal{H}(P_A \| P) = \int 1_A \log\left(\frac{1}{P(A)}\right) dP_A = -\log P(A)$, and thus the maximal expected *logarithmic* utility relative to P_A is given by

$$u_A(x) = \log(x) - \log P(A).$$

In order to determine the power utility, let $p \in (0,1)$, and observe that $\left(\int \left(\frac{dP_A}{dP}\right)^{\frac{1}{1-p}} dP\right)^{1-p} = \left(\int 1_A \left(\frac{1}{P(A)}\right)^{\frac{1}{1-p}} dP\right)^{1-p} = \frac{1}{P(A)^p}$. Therefore, the maximal expected *power* utility relative to p is given by

$$u_A(x) = \frac{1}{p}\left(\frac{x}{P(A)}\right)^p.$$

4 Enlarging filtrations with finite partitions

Throughout this section let $\mathcal{P} = \{A_1, \ldots, A_n\}$ be a finite partition of Ω into $\mathcal{F}$-measurable sets. We will study the utility maximisation problem under the enlarged filtration

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(\mathcal{P}), \; t \ge 0.$$

Recall that the maximal expected utility is given by $u^{\mathcal{G}}(x) = \sup\{EU(x + (\theta \cdot S)_T) : \theta \in \mathcal{A}(\mathcal{G}) \text{ is } x-\text{admissible}\}$.

Definition 4.1. The *conditional maximal expected utility* relative to the partition $\mathcal{P}$ is defined by

$$Z(x) = \sum_{i=1}^n 1_{A_i} u_{A_i}(x), \quad x > 0.$$

Lemma 4.2.

$$EZ(x) = u^{\mathcal{G}}(x).$$

Proof. Let $\theta \in \mathcal{A}(\mathcal{G})$. Observe that $\theta^i = 1_{A_i}\theta \in \mathcal{A}(\mathcal{G})$, and $1_{A_i}(\theta \cdot S)_t = (\theta^i \cdot S)_t$. Therefore,

$$\begin{aligned} EU(x + (\theta \cdot S)_T) &= \sum_{i=1}^n P(A_i) E[U(x + (\theta^i \cdot S)_T) | A_i] \\ &\le \sum_{i=1}^n P(A_i) u_{A_i}(x), \end{aligned}$$

and thus $u^{\mathcal{G}}(x)$ is smaller than the LHS. For the reverse inequality, observe that Lemma 3.2 implies that the maximal expected utility $u_{A_i}(x)$ can be attained by using strategies in $\mathcal{A}(\mathcal{F})$. Consequently, for every $\varepsilon > 0$ and i, we may choose $\theta^i \in \mathcal{A}(\mathcal{F})$ such that $E[U(x+(\theta^i \cdot S)_T)|A_i] \geq u_{A_i}(x) - \varepsilon$. Then, $\theta = \sum_{i=1}^n 1_{A_i}\theta^i \in \mathcal{A}(\mathcal{G})$, and $EU(x + (\theta \cdot S)_T) \geq -\varepsilon + \sum_i P(A_i)u_{A_i}(x)$. Hence the proof is complete. □

The results of the preceding section imply that the stochastic convex conjugate of $-Z(-x)$, defined by $Y(y) = \sup_{x>0}[Z(x) - xy]$ $(y > 0)$, is given by

$$Y(y) = \sum_{i=1}^n 1_{A_i} v_{A_i}(y) = \sum_{i=1}^n 1_{A_i} (V_y)\hat{}(P_{A_i}\|Q).$$

For later use we rewrite Y as a divergence of two measures on the product space $\bar{\Omega} = \Omega \times \Omega$. The first measure is the image of P under the embedding $\psi : \Omega \to \bar{\Omega}, \omega \mapsto (\omega, \omega)$. We put $\bar{P} = P \circ \psi^{-1}$. The other measure is the product $\bar{Q} = Q \otimes P$.

Lemma 4.3. *Let $\mathcal{A} = \mathcal{F}_T \otimes \sigma(\mathcal{P})$. Then*

$$Y(y) = \int (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}\Big|_{\mathcal{A}}\right) dQ, \quad y > 0.$$

Proof. Note that $\frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{A}}(\omega, \omega') = \sum_{i=1}^n \frac{dP_{A_i}}{dQ}(\omega)\, 1_{A_i}(\omega')$, and hence

$$\begin{aligned}
\int (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}\right)(\omega, \omega')\, dQ(\omega) &= \int (V_y)\hat{}\left(\sum_{i=1}^n \frac{dP_{A_i}}{dQ}(\omega)\, 1_{A_i}(\omega')\right) dQ(\omega) \\
&= \sum_{i=1}^n 1_{A_i}(\omega') \int (V_y)\hat{}\left(\frac{dP_{A_i}}{dQ}\right) dQ(\omega) \\
&= \sum_{i=1}^n 1_{A_i}(\omega') v_{A_i}(y) \\
&= Y(y)(\omega').
\end{aligned}$$

□

Remark 4.4. Suppose that we have an enlargement $\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(G)$, $t \geq 0$, where G is an arbitrary random variable with values in a polish space Γ. An analogue of Lemma 4.2 is true for the expected utility conditioned on G. One could try to prove this directly with the same idea, however, one would have to show tedious details. Among other things, we would have to check measurability and S-integrability of processes $(\omega, t) \mapsto \theta_t^{G(\omega)}(\omega)$, where $\theta^g \in \mathcal{A}(\mathcal{F})$ for all $g \in \Gamma$. Instead of doing so, we derive the generalisation with the help of a monotone convergence property which is interesting for its own.

5 Monotone utility convergence

Let $(\mathcal{H}_t^n)$ be an increasing sequence of filtrations with usual conditions and such that $\mathcal{H}_t^n \supset \mathcal{F}_t$. Moreover, let

$$\mathcal{H}_t = \bigcap_{s>t} \sigma\left(\mathcal{H}_s^n : n \geq 1\right).$$

Throughout we assume that the price process S is a continous semimartingale with respect to all filtrations $(\mathcal{F}_t)$, $(\mathcal{H}_t^n)$ and $(\mathcal{H}_t)$. Moreover, we suppose that the maximal expected utility $u^{\mathcal{H}}(x)$ relative to $(\mathcal{H}_t)$ is finite, and that there exists a $(\mathcal{H}_t)$-predictable process such that the Doob-Meyer decomposition is given by

$$S_t = S_0 + M_t + \int_0^t \alpha_s d\langle M, M\rangle_s, \quad t \geq 0,$$

If $\lim_{x\to\infty} U(x) = \infty$, then the existence of such a decomposition follows already from the finiteness of $u^{\mathcal{H}}(x)$ (see [3]). Since $(\mathcal{H}_t^n)$ is increasing, the sequence of the related maximal expected utility is also increasing. Indeed, as will be shown below, it satisfies a monotone convergence property. We will apply this result later in order to prove Theorem 2.6. Note that we do not assume here that $(\mathcal{H}_t^n)$ is an initial enlargement of $(\mathcal{F}_t)$.

To simplify notation, let $u^n(x)$ denote the maximal expected utility with respect to $(\mathcal{H}_t^n)$. We start with the observation that the utility maximum can be attained by L^2-integrable strategies. Given a filtration $(\mathcal{G}_t)$, we denote by $L^2_{\mathcal{G}}(M)$ the set of all $(\mathcal{G}_t)$-predictable processes θ such that $E\int_0^T \theta_s^2 d\langle M, M\rangle_s < \infty$.

Lemma 5.1. *Let $x > 0$. Then*

$$\begin{aligned} u^{\mathcal{G}}(x) &= \sup\{EU(x + (\theta\cdot S)_T) : \theta \in L^2_{\mathcal{G}}(M) \cap \mathcal{A}(\mathcal{G}), \\ &\qquad (x-\varepsilon) - \textit{adm. for some } \varepsilon > 0\}. \end{aligned} \tag{4}$$

Proof. We prove at first for all $x > 0$

$$u^{\mathcal{G}}(x) = \sup_{\varepsilon>0} \sup\{EU(x + (\theta\cdot S)_T) : \theta \in \mathcal{A}(\mathcal{G}),\ (x-\varepsilon) - \text{adm.}\} \tag{5}$$

We have only to show that the LHS does not exceed the RHS. For this let $\theta \in \mathcal{A}(\mathcal{G})$ such that $EU(x + (\theta\cdot S)_T) > -\infty$. Put $\theta^n = (1-\frac{1}{n})\theta$ for all $n \geq 1$. Clearly, θ^n is $(x - \frac{x}{n})$-admissible. Monotone convergence applied to the negative and positive part of $U(x + (\theta^n\cdot S)_T) - U(x)$ implies

$$\lim_n EU(x + (\theta^n \cdot S)_T) = EU(x + (\theta\cdot S)_T),$$

and hence $u^{\mathcal{G}}(x)$ is smaller than RHS of equation (5).

In equation (4) the RHS is obviously not greater than the LHS. For the reverse inequality choose $\varepsilon > 0$ and an $(x-\varepsilon)$-admissible strategy θ satisfying $EU(x + (\theta \cdot S)_T) > -\infty$. Due to (5) it is sufficient to show that $EU(x + (\theta \cdot S)_T)$ is not bigger than the RHS of (4). Since θ is S-integrable, the stopping times

$$T_n = T \wedge \inf\{t \geq 0 : \int_0^t \theta_r^2 \, d\langle M, M\rangle_r \leq n\}$$

converge almost surely to T for $n \to \infty$. Note that the strategies $\theta^n = 1_{[0,T_n]}\theta$ are $(x-\varepsilon)$-admissible and belong to $L^2_{\mathcal{G}}(M)$. Fatou's Lemma implies

$$\liminf_n EU(x + (\theta^n \cdot S)_T) \geq EU(x + (\theta \cdot S)_T),$$

and thus the result. □

Here is the main theorem of this section.

Theorem 5.2. *Let $x > 0$. Then*

$$\lim_n u^{\mathcal{H}^n}(x) = u^{\mathcal{H}}(x).$$

Proof. Let $\theta \in L^2_{\mathcal{H}}(M)$ be $(x-\varepsilon)$-admissible. The stopping times

$$\tau_k = T \wedge \inf\{t \geq 0 : \int_0^t \alpha_s^2 \, d\langle M, M\rangle_s \geq k\}$$

converge to T, a.s, and hence

$$\liminf_k EU(x + (\theta \cdot S)_{\tau_k}) \geq EU(x + (\theta \cdot S)_T).$$

By Lemma 5.1 it suffices to show that for all $k \geq 1$, $EU(x + (\theta \cdot S)_{\tau_k})$ is not greater than $\sup_n u^{\mathcal{H}^n}(x)$. To simplify notation we assume that $\tau_k = T$ for some k.

Now let θ^n be the projection of θ onto $L^2_{\mathcal{H}^n}$. By Doob's inequality there is a constant $C > 0$, such that

$$\begin{aligned}
E((\theta^n - \theta) \cdot S)^*_T &\leq E((\theta^n - \theta) \cdot M)^*_T + E((\theta^n - \theta)\alpha \cdot \langle M, M\rangle)^*_T \\
&\leq \|((\theta^n - \theta) \cdot M)^*_T\|_2 + E((\theta^n - \theta)\alpha \cdot \langle M, M\rangle)^*_T \\
&\leq C\,\|((\theta^n - \theta) \cdot M)_T\|_2 + E(|\theta^n - \theta||\alpha| \cdot \langle M, M\rangle)_T.
\end{aligned}$$

The first summand in the preceding line goes to 0, because (θ^n) converges to θ in $L^2_{\mathcal{H}}(M)$. The second vanishes due to Kunita-Watanabe and due to our assumption that $\int_0^T \alpha_s^2 \, d\langle M, M\rangle_s$ is bounded. Consequently, by choosing a subsequence if necessary, almost everywhere the sequence $(\theta^n \cdot S)$ converges uniformly to $(\theta \cdot S)$ on $[0,T]$. Now put

$$T_n = T \wedge \inf\{t \geq 0 : (\theta^n \cdot S)_t \leq -x + \frac{\varepsilon}{2}\}$$

and

$$\pi^n = 1_{[0,T_n]}\theta^n.$$

The strategies π^n are $(x - \frac{\varepsilon}{2})$-admissible and satisfy almost surely

$$\lim_n (\pi^n \cdot S)_T = (\theta \cdot S)_T.$$

With Fatou's Lemma we obtain

$$\liminf_n EU(x + (\pi^n \cdot S)_T) \geq EU(x + (\theta \cdot S)_T),$$

and hence the result. □

We close this section with an example showing that Theorem 5.2 is not valid without the assumption that S is continuous.

Example 5.3. Let $T > 1$ and ϕ a standard normal random variable. Suppose the price process is given by

$$S_t = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ 1 + \phi + \frac{1}{2}, & \text{if } 1 \leq t \leq T, \end{cases}$$

and let $(\mathcal{F}_t^S)$ be the completed filtration generated by S. Moreover let (ε_n) be a sequence of independent normal random variables with mean zero and $\mathrm{Var}(\varepsilon_n) = \frac{1}{n}$. We define

$$\mathcal{H}_t^n = \mathcal{F}_t \vee \sigma(1_{\{|\phi| \geq 1\}} + \varepsilon_n), \quad 0 \leq t \leq T,$$

and claim that

$$u^{\mathcal{H}^n}(x) = U(x)$$

for all $x > 0$. For this let θ be $(\mathcal{H}_t^n)$-predictable and S-integrable. If $\theta_1 \neq 0$ a.s., then the integral $(\theta \cdot S)_1$ is unbounded from below and hence θ is not admissible. Since the process S is constant on the remaining part of the trading interval, we have $u_a^{\mathcal{H}^n}(x) = U(x)$. A trader having access to

$$\mathcal{H}_t = \bigvee_{n \geq 1} \mathcal{H}_t^n$$

knows whether the absolute value of ϕ is bigger or smaller than 1. Therefore he has access to non-trivial admissible trading strategies. As a consequence $u^{\mathcal{F}}(x) > U(x)$, and hence

$$\lim_n u^{\mathcal{H}^n}(x) \neq u^{\mathcal{H}}(x).$$

By the way, the price process S satisfies the (NFLVR) property with respect to $(\mathcal{H}_t)$.

6 General initial enlargements

Let us now consider investors with additional information represented by an enlarged filtration

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(G),\ t \geq 0,$$

where G is an arbitrary random variable with values in a polish space Γ. In Section 4 we considered filtrations enlarged by a finite partition. An approximation through discretisation will allow us to generalise the results.

We will write the dual of the conditional expectation as a reverse V_y-divergence of the two measures $\bar{P}$ and $\bar{Q}$ on the product space $\bar{\Omega} = \Omega \times \Omega$. Recall that $\bar{P}$ is the image of P under the embedding $\psi : \omega \mapsto (\omega, \omega)$, and $\bar{Q} = Q \otimes P$.

Note that the embedded price process $\bar{S}(\omega, \omega') = S(\omega)$ is a $\bar{Q}$-local martingale with respect to the filtration $\bar{\mathcal{F}}_t = \bigcap_{s>t} \mathcal{F}_s \otimes \sigma(G)$. The product measure $\bar{Q}$ plays the role Q had in the old setting.

We abbreviate $\mathcal{A} = \mathcal{F}_T \otimes \sigma(G)$. Throughout we assume that $(V_y)\hat{}_{\mathcal{A}} \left(\bar{P} \| \bar{Q}\right)$ is finite for all $y > 0$. In particular, $\bar{P} \ll \bar{Q}$ on $\mathcal{A}$.

Let $(\mathcal{P}^n)$ be a sequence of finite partitions of $\sigma(G)$ such that $\sigma(G) = \bigvee_n \sigma(\mathcal{P}^n)$, and $\mathcal{P}^n \subset \mathcal{P}^{n+1}$. We denote by $u^n(x)$ the maximal expected utility under the filtration enlarged by $\sigma(\mathcal{P}^n)$. Then monotone utility convergence implies that $\lim_n u^n(x) = u^{\mathcal{G}}(x)$. As before, for all $A \in \mathcal{P}^n$, we denote by $u_A(x)$ the maximal expected utility under P_A and the filtration enlarged by $\sigma(\mathcal{P}^n)$. Again, let $Z_n(x) = \sum_{A \in \mathcal{P}^n} 1_A\, u_A(x)$ be the conditional expected utility with respect to $\mathcal{P}^n$.

Lemma 6.1. *Let $x > 0$. Then $(Z_n(x))$ is a submartingale with respect to the filtration $\mathcal{H}_n = \sigma(\mathcal{P}^n)$, $n \geq 1$. Moreover, $(Z_n(x))$ is uniformly integrable and convergent in L^1.*

Proof. Let $n \geq 1$, and $A \in \mathcal{P}^n$. Since $\mathcal{P}^n \subset \mathcal{P}^{n+1}$, there are sets $B_1, \ldots, B_k$ in $\mathcal{P}^{n+1}$ such that $A = B_1 \cup \ldots \cup B_k$. Obviously

$$\begin{aligned} E[1_A Z_n(x)] &= P(A) u_A(x) \\ &\leq P(B_1) u_{B_1}(x) + \ldots + P(B_k) u_{B_k}(x) \\ &= E[1_A Z_{n+1}(x)]. \end{aligned}$$

Therefore $E[Z_{n+1}(x) | \mathcal{H}_n] \geq Z_n(x)$, for all $n \geq 1$, which means that $(Z_n(x))$ is a submartingale.

Note that $u_A(x) \geq U(x)$, and hence $Z_n(x) \geq U(x)$, a.s. Assume for simplicity that $U(x) \geq 0$ (else consider $Z_n(x) - U(x)$). Then $\sum_{C \in \mathcal{C}} P(A)\, u_A(x) \leq$

$u^{\mathcal{G}}(x) < \infty$, for every subset $\mathcal{C} \subset \mathcal{P}^n$. In particular

$$E\left[Z_n(x); Z_n(x) \geq M\right] \leq u^{\mathcal{G}}(x),$$

showing that the submartingale $(Z_n(x))$ is uniformly integrable. As a consequence, it converges in L^1 (see f.e. Chapter 4 in [9]). □

Definition 6.2. The L^1-limit $Z(x) = \lim_n Z_n(x)$ will be called *conditional expected utility* relative to G. Note that $Z(x)$ is $\sigma(G)$-measurable, and for all $B \in \sigma(G)$ with $P(B) > 0$, we have $u_B(x) = \int_B Z(x)\, dP$.

We apply now the results of Section 4 to our approximations $Z_n(x)$: Lemma 4.3 implies that the stochastic conjugate of $Z_n(x)$ is given by

$$Y_n(y) = \int (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T \otimes \sigma(\mathcal{P}^n)}\right)(\omega, \omega')\, dQ(\omega), \quad y > 0,$$

P-almost surely. We claim that $Y_n(y)$ converges to

$$Y(y) = \int (V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T \otimes \sigma(G)}\right)(\omega, \omega')\, dQ(\omega).$$

More precisely:

Lemma 6.3. *The processes $Y(y)$ and $Y_n(y)$ are strictly convex, decreasing and differentiable on $(0, \infty)$, almost surely. The derivatives satisfy*

$$\lim_{y \downarrow 0} Y'(y) = -\infty, \text{ and } \lim_{y \to \infty} Y'(y) = 0.$$

Moreover, for almost all ω the functions $y \mapsto Y_n(y)$ converge pointwise to $y \mapsto Y(y)$ on $(0, \infty)$, and the derivatives $Y_n'(y)$ converge to the derivative $Y'(y)$.

Proof. With similar arguments as in Lemma 3.4 one can show the first and second statement. For the third, let $y > 0$. Note that $(V_y)\hat{}_{\mathcal{F}_T \otimes \sigma(\mathcal{P}^n)}\left(\bar{P} \| \bar{Q}\right)$ converges to $(V_y)\hat{}_{\mathcal{A}}\left(\bar{P} \| \bar{Q}\right)$ (see Lemma A.1). Assume that $(V_y)\hat{}$ is strictly positive, else choose a modification as in Lemma A.2. As a consequence, the sequence $(V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T \otimes \sigma(\mathcal{P}^n)}\right)$ converges to $(V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{A}}\right)$ in $L^1(\bar{Q})$. Moreover, $(V_y)\hat{}\left(\frac{d\bar{P}}{d\bar{Q}}|_{\mathcal{F}_T \otimes \sigma(\mathcal{P}^n)}\right)$ is a uniformly integrable submartingale relative to $\bar{Q}$, and therefore it converges $\bar{Q}$-almost surely. By Fubini's theorem, $(Y_n(y))$ converges to $Y(y)$ for almost all ω.

We now apply the following well-known fact from convex analysis: If (f_n) is sequence of convex functions converging on a dense subset to a finite function f, then (f_n) converges to f everywhere and f is convex. Moreover (f_n) converges uniformly on every bounded set (see Theorem 10.8 in [20]).

In order to apply this result to our processes (Y_n), let $\mathcal{D}$ be a countable dense subset of $(0, \infty)$. For any $q \in \mathcal{D}$ we have $\lim_n Y_n(q) = Y(q)$ almost surely, and therefore, for almost all ω, the functions $y \mapsto Y_n(y)$ converge pointwise to $y \mapsto Y(y)$ on $(0, \infty)$.

Finally, another result from convex analysis implies that the derivatives converge almost surely, i.e. $\lim_n Y_n'(y) = Y'(y)$ (see Theorem 25.7 in [20]). □

Due to the previous lemma we may define

$$W_n(z) = \left(\frac{d}{dy} Y_n\right)^{-1}(z), \text{ and } W(z) = \left(\frac{d}{dy} Y\right)^{-1}(z), \quad z < 0.$$

Note that Lemma 6.3 implies $\lim_n W_n(z) = W(z)$, a.s.

Theorem 6.4. *For almost all ω, the two processes (Z, Y) are a Legendre pair. Moreover, $u^{\mathcal{G}}(x) = E[xW(-x) + Y(W(-x))]$, for $x > 0$.*

Proof. We have to show that $Z(x, \omega) = xW(-x, \omega) + Y(W(-x, \omega), \omega)$ for almost all ω. According to Theorem 2.6

$$Z_n(x, \omega) = xW_n(-x, \omega) + Y_n(W_n(-x, \omega), \omega),$$

almost surely. Moreover, by Lemma 6.3, for almost all ω the functions $y \mapsto Y_n(y)$ converge pointwise to $y \mapsto Y(y)$ on $(0, \infty)$. Since these functions converge uniformly on every closed bounded subset of $(0, \infty)$, we have

$$\lim_n Y_n(W_n(x, \omega), \omega) = Y(W(x, \omega), \omega),$$

and hence $Z(x) = xW(x) + Y(W(x))$, almost surely. This shows that Z and Y are dual. The properties of Y imply that $\lim_{x \downarrow 0} Z(x) = \infty$, and $\lim_{x \to \infty} Z(x) = 0$, almost surely. Finally, the last statement follows from $u^{\mathcal{G}}(x) = E[Z(x)]$. □

Proof to Theorem 2.6. Let π be a regular conditional probability of P with respect to $\sigma(G)$. Then, on $\mathcal{A}$, we have

$$\frac{\pi(d\omega, \omega')}{dQ(\omega)} = \frac{d\bar{P}}{d\bar{Q}}(\omega, \omega').$$

Therefore, Theorem 2.6 is just a reformulation of Theorem 6.4. □

$\bar{Q}$ as a martingale measure

Under the product measure $\bar{Q}$ the *new* information $\sigma(G)$ is independent of the *old* information $\mathcal{F}_T$. Therefore, $\bar{Q}$ can be referred to as a *decoupling measure*. This measure was used in [3] in order to derive, with the Girsanov-Lenglart Theorem, semimartingale decompositions relative to $(\mathcal{G}_t)$. If the conditional probabilities $\pi(\cdot, \omega)$ are equivalent to P, then there exist measures on the *original space* which decouple the new from the old information. The utility maxisation problem in this case has been considered in [1].

The measure $\bar{Q}$ is not the only local martingale measure of the embedded process $\bar{S}(\omega, \omega') = S(\omega)$ with respect to the filtration $(\bar{\mathcal{F}}_t)$: if $P' \sim P$ on $\sigma(G)$, then $R \otimes P'$ is a martingale measure equivalent to $\bar{Q}$. However, among all these measures $R \otimes P'$, the measure $\bar{Q}$ minimises the entropy relative to $\bar{P}$.

In order to sketch the proof, let $\mathcal{P}$ denote the set of probability measures P' equivalent to P and defined on $\sigma(G)$. Then $R \otimes P' \sim \bar{Q}$ and, since $\bar{P} \ll R \otimes P$, we also have $\bar{P} \ll R \otimes P'$ for all $P' \in \mathcal{P}$. Note that

$$\begin{aligned} \inf_{P' \in \mathcal{P}} \mathcal{H}(\bar{P} \| R \otimes P') &= \inf_{P' \in \mathcal{P}} \int \log\left(\frac{d\bar{P}}{dR \otimes P}\right) + \log\left(\frac{dR \otimes P}{dR \otimes P'}\right) \, d\bar{P}, \\ &= \mathcal{H}(\bar{P} \| R \otimes P) + \inf_{P' \in \mathcal{P}} \int \log\left(\frac{dP}{dP'}\right) \, dP. \end{aligned}$$

It is straightforward to show that the right hand side attains its infimum if $P' = P$, and hence $\bar{Q}$ is the *entropy minimising martingale measure* for $\bar{S}$ with respect to the enlarged filtration $(\bar{\mathcal{F}}_t)$.

We close again the section by considering special utility functions.

Proposition 6.5. *(Logarithmic utility)*
Let $U(x) = \log x$ for all $x > 0$. Then $W(-x) = \frac{1}{x}$ and the maximal expected utility is equal to the relative entropy of $\bar{P}$ with respect to $\bar{Q}$, i.e. $u^{\mathcal{G}}(x) = \log(x) + \mathcal{H}(\bar{P} \| \bar{Q})$.

Proof. It is straightforward to show that the conjugate is given by $Y(y) = -\log(y) - 1 + \int \log \frac{d\bar{P}}{d\bar{Q}} \, dQ$, and $W(z) = -\frac{1}{z}$. The claim follows from Theorem 6.4. □

Proposition 6.6. *(Power utility)*
Let $0 < p < 1$ and $U(x) = \frac{1}{p} x^p$ for all $x \geq 0$. Then

$$u^{\mathcal{G}}(x) = U(x) \int \left(\int \left(\frac{d\bar{P}}{d\bar{Q}} \right)^{\frac{1}{1-p}} dQ \right)^{1-p} dP.$$

Proof. Recall that $(V_y)\hat{}(x) = \left(\frac{1-p}{p}\, y^{-\frac{p}{1-p}}\right) x^{\frac{1}{1-p}}$. The stochastic conjugate Y of the conditional expected utility Z satisfies

$$Y(y) = \frac{1-p}{p}\, y^{-\frac{p}{1-p}} \int \left(\frac{d\bar{P}}{d\bar{Q}}\right)^{\frac{1}{1-p}} dQ.$$

Moreover, $W(z) = (-z)^{-(1-p)} \left(\int \left(\frac{d\bar{P}}{d\bar{Q}}\right)^{\frac{1}{1-p}} dQ\right)^{1-p}$, $z < 0$, and with Theorem 6.4, the result. □

Remark 6.7. For the logarithm we need not to consider the *conditional* expected utility, in order to generate the conjugate function of $u^{\mathcal{G}}(x)$. Recall $Z(x) = \inf_{y>0}[xy + Y(y)]$. By taking expectations we obtain

$$EZ(x) = \mathcal{H}(\bar{P}\|\bar{Q}) - 1 + \inf_{y>0}[xy - \log(y)],$$

and hence the conjugate of $u^{\mathcal{G}}(x)$ is given by

$$v^{\mathcal{G}}(y) = -\log(y) + \mathcal{H}(\bar{P}\|\bar{Q}) - 1.$$

7 Additional logarithmic utility

The properties of the logarithm lead to simple formulas for the additional logarithmic utility of an investor with information $\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \vee \sigma(G)$ compared to an investor having only access to $(\mathcal{F}_t)$.

Theorem 7.1. *If* $U = \log$, *then the utility difference* $\Delta u = u^{\mathcal{G}}(x) - u^{\mathcal{F}}(x)$ *does not depend on* x, *and it is equal to the* mutual information *between* $\mathcal{F}_T$ *and* G, *i.e.*

$$\Delta u = \mathcal{H}_{\mathcal{F}_T \otimes \sigma(G)}(\bar{P}\,\|P \otimes P) = I(\mathcal{F}_T, G).$$

In particular, if G *is discrete and* $\mathcal{F}_T$*-measurable, the additional utility is equal to the absolute entropy of* G *relative to* P,

$$\Delta u = -\sum_g P(G = g) \log P(G = g).$$

Proof. Let $f(\omega, \omega') = \frac{d\bar{P}}{d\bar{Q}}\big|_{\mathcal{F}_T \otimes \sigma(G)}$ and $g(\omega) = \frac{dQ}{dP}\big|_{\mathcal{F}_T}$. We show at first that

$$f(\omega, \omega') g(\omega) = \frac{d\bar{P}}{d(P \otimes P)}\big|_{\mathcal{F}_T \otimes \sigma(G)}. \tag{6}$$

For this let $A \in \mathcal{F}_T$ and $B \in \sigma(G)$. Note that

$$\begin{aligned}
\int 1_{A\times B}(\omega,\omega')f(\omega,\omega')g(\omega)\,d(P\otimes P) &= \int 1_{A\times B}(\omega,\omega')f(\omega,\omega')\,d(Q\otimes P)\\
&= \int 1_{A\times B}(\omega,\omega')f(\omega,\omega')\,d\bar{Q}\\
&= \bar{P}(A\times B),
\end{aligned}$$

which implies (6). Now recall that $u^{\mathcal{G}}(x) = \log(x) + \mathcal{H}(\bar{P}\|\bar{Q})$ and $u^{\mathcal{F}}(x) = \log(x) + \mathcal{H}(P\|Q)$. Thus

$$\begin{aligned}
u^{\mathcal{G}}(x) - u^{\mathcal{F}}(x) &= \mathcal{H}(\bar{P}\|\bar{Q}) - \mathcal{H}(P\|Q)\\
&= \int \left(\log f(\omega,\omega') - \log g^{-1}(\omega)\right)\,d\bar{P}\\
&= \int \log\left(f(\omega,\omega')g(\omega)\right)\,d\bar{P}\\
&= \int \log\left(\frac{d\bar{P}}{d(P\otimes P)}\Big|_{\mathcal{F}_T\otimes\sigma(G)}\right)\,d\bar{P}\\
&= \mathcal{H}_{\mathcal{F}_T\otimes\sigma(G)}(\bar{P}\,\|P\otimes P).
\end{aligned}$$

Finally, if G is discrete and $\mathcal{F}_T$-measurable, then Δu is equal to the absolute entropy of G. □

Remark 7.2. Let $\mathcal{H}_t = \bigcap_{s>t}\mathcal{F}_s \vee \sigma(H)$ be another initially enlarged filtration such that $\sigma(H)$ is a sub-σ-field of $\sigma(G)$. Then the logarithmic utility difference $u^{\mathcal{G}} - u^{\mathcal{H}}$ is equal to the mutual information of $\mathcal{F}_T$ and G *conditioned* on H, which is defined by $I(\mathcal{F}_T, G|H) = I(\mathcal{F}_T, G) - I(\mathcal{F}_T, G)$ (see [11]).

Maximal expected utility for non-initial enlargements

So far we considered *initial* enlargements of a given filtration and we determined the conjugate function of the maximal expected utility conditioned on the enlarging random variable. What can we do, if the filtration is not only enlarged in the beginning, but at any moment during the trading period? Can we still determine a conjugate of the maximal expected utility?

Due to monotone utility convergence we can approximate general enlargements by piecewise initial enlargements of the filtration: the trading interval is devided into small subintervals, and in the beginning of each subinterval the filtration is enlarged initially. Naturally, the idea arises to apply our results to each subinterval, and thus derive again a dual representations of the maximal expected utility via f-divergences. Unfortunately there is the following problem: Let t be a point in the trading interval $(0, T)$. The maximal utility up to time T is in general *not* the sum of the maximal utility up to time t and the maximal utility between t and T. It is shown in [5] that the logarithm is essentially the only utility function to have this property. So let for the remaining section $U = \log$.

Let $(\mathcal{H}_t)$ be a filtration. We will maximise logarithmic utility with respect to the enlarged filtration

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{H}_s).$$

We approximate $(\mathcal{G}_t)$ with piecewise initially enlarged filtrations. For this let for any partition $\Delta: \ 0 = t_0 \leq \ldots \leq t_n = T$,

$$\mathcal{G}_t^{\Delta} = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{H}_{t_i}) \quad \text{if } t \in [t_i, t_{i+1}[.$$

According to Remark 7.2, the additional logarithmic utility relative to $(\mathcal{G}_t^{\Delta})$ is given by $u^{\Delta}(x) = I(\mathcal{F}_T, \mathcal{H}_{t_0}) + \sum_{i=0}^{n-1} I(\mathcal{F}_T, \mathcal{H}_{t_{i+1}} | \mathcal{H}_{t_i}, \mathcal{F}_{t_{i+1}})$. Now let (Δ_n) be a sequence of partitions such that $\Delta_n \subset \Delta_{n+1}$ and $\lim_n |\Delta_n| = 0$. Monotone utility convergence implies $u^{\mathcal{G}}(x) = \lim_{n\to\infty} u^{\Delta_n}(x)$. As a consequence, the sums $\left(I(\mathcal{F}_T, \mathcal{H}_{t_0}) + \sum_{i=0}^{n-1} I(\mathcal{F}_T, \mathcal{H}_{t_{i+1}} | \mathcal{H}_{t_i}, \mathcal{F}_{t_{i+1}}) \right)$ converge, and we denote the limit as the *mutual information* between the filtrations $(\mathcal{F}_t)$ and $(\mathcal{H}_t)$. We have thus shown the following result.

Theorem 7.3. *Let $U = \log$. Then the logarithmic utility difference $u^{\mathcal{G}} - u^{\mathcal{F}}$ is equal to the mutual information between $(\mathcal{F}_t)$ and $(\mathcal{H}_t)$, i.e.*

$$u^{\mathcal{G}} - u^{\mathcal{F}} = \lim_n \left(I(\mathcal{F}_T, \mathcal{H}_{t_0}) + \sum_{i=0}^{n-1} I(\mathcal{F}_T, \mathcal{H}_{t_{i+1}} | \mathcal{H}_{t_i}, \mathcal{F}_{t_{i+1}}) \right).$$

A similar result has already been derived in [4], however, in a completely different way.

A Appendix: f-Divergences

Lemma A.1. *(see Theorem 1.30 in [17])*
Let $(\mathcal{A}_n)$ be a sequence of increasing sub-σ-fields and $\mathcal{A} = \bigvee_n \mathcal{A}_n$. Then $(f_{\mathcal{A}_n}(P\|Q))$ is an increasing sequence and

$$\lim_n f_{\mathcal{A}_n}(P\|Q) = f_{\mathcal{A}}(P\|Q).$$

Lemma A.2. *Let f be a finite convex function on $(0, \infty)$. Then there exists a non-negative and convex function g and a constant $C \in \mathbb{R}$ such that for all $P \ll Q$ we have $g(P\|Q) = f(P\|Q) + C$.*

Proof. If f is bounded from below, then put $g(x) = f(x) - C$, where C is the minimal value of f.

Now suppose that $\lim_{x\to\infty} f(x) = -\infty$. Then $f(x) \geq f(1) + f'(1)(x-1)$, for all $x > 0$. Therefore, $g(x) = f(x) - f(1) - f'(1)(x-1)$ is a non-negative convex function. Moreover, for all $P \ll Q$, we have

$$\begin{aligned} g(P\|Q) &= \int g\left(\frac{dP}{dQ}\right) dQ = -f(1) + \int f\left(\frac{dP}{dQ}\right) dQ - f'(1) \int \left(\frac{dP}{dQ} - 1\right) dQ \\ &= -f(1) + f(P\|Q), \end{aligned}$$

and thus the result. $\square$

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