Diffusion control ranking games
Part III
The role of correlation in the 2 player game

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AIMS Ghana, Accra, December 2024

Recall the 2-player game

- State of player 1:

$$dX_t = \alpha(X_t, Y_t)dW_t^1, \quad X_0 = 0$$

- State of player 2:

$$dY_t = \beta(X_t, Y_t)dW_t^2, \quad Y_0 = 0$$

- $\alpha, \beta : \mathbb{R}^2 \to [\sigma_1, \sigma_2]$ **'strict controls'**

$$\text{reward of player 1} = \begin{cases} 1, & \text{if } X_T > Y_T, \\ 0, & \text{else.} \end{cases}$$

$$\text{reward of player 2} = \begin{cases} 1, & \text{if } Y_T > X_T, \\ 0, & \text{else.} \end{cases}$$

Now: BM are correlated

$$dX_t = \alpha(X_t, Y_t)dW_t^1$$
$$dY_t = \beta(X_t, Y_t)dW_t^2$$

New assumption: W^1 and W^2 are BM with constant correlation

$$\rho = \text{Corr}(W_t^1, W_t^2).$$

- So far we have assumed $\rho = 0$.

Why does the correlation have an impact?

Recall that if $\rho = 0$, then (α^*, β^*) with

$$\alpha^*(x,y) = \begin{cases} \sigma_1, & \text{if } x \geq y, \\ \sigma_2, & \text{if } x < y, \end{cases}$$

and

$$\beta^*(x,y) = \alpha^*(y,x).$$

is an equilibrium.

If $\rho = 1$ this can not be an equilibrium!

Case: $\rho = 1$

In this case $D_t := X_t - Y_t$ satisfies

$$dD_t = (\alpha_t - \beta_t)dW_t^1$$

- If ahead, player 1 wants to choose $\alpha_t = \beta_t$.
- If behind, player 1 wants to choose α_t as far away from β_t as possible.

$\Longrightarrow$ **There is no equilibrium in strict controls**

Questions

1. Up to which correlation threshold does there exist an equilibrium in strict controls?
2. Can we define mixed strategies so that an equilibrium always exists?

The correlation threshold

Theorem

*The game has a value in strict controls **if and only if***

$$\rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}. \tag{1}$$

In this case the value function is given by

$$V_{strict}(t, x, y) = \Phi\left(\frac{x - y}{c(\rho)\sqrt{T - t}}\right), \qquad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},$$

and a saddle point is given by

$$\alpha^*(x, y) = \begin{cases} \sigma_2, & \text{if } x \leq y, \\ \sigma_1 \vee \rho\sigma_2, & \text{if } x > y, \end{cases}$$

$$\beta^*(x, y) = \begin{cases} \sigma_1 \vee \rho\sigma_2, & \text{if } x \leq y, \\ \sigma_2, & \text{if } x > y. \end{cases}$$

What is the right notion of a mixed strategy in differential games?

1st attempt: randomize continuously

Problem: If $(\alpha_t)_{t\in[0,1]}$ is iid, then $(\omega, t) \mapsto \alpha_t(\omega)$ is not measurable!

A proof can be found here:
`https://math.stackexchange.com/questions/4271985/example-of-non-measurable-stochastic-process`

2nd attempt: discretize and take limits

$$\alpha_t^n = \xi_k \qquad \text{for } t \in \left[\frac{k}{n}T, \frac{k+1}{n}\right)$$

where (ξ_k) is iid with $\sim \mu$.

Question: Where does α^n converge to?

Caution: α^n does not converge in a process space

Idea: Embed α^n into the space of **probability measures** on $[\sigma_1, \sigma_2] \times [0, T]$. The measure $\delta_{\alpha_t^n}(da)dt$ converges weakly to

$$\mu(da)dt.$$

Relaxed controls

Definition
A relaxed (Markov) control is a measurable mapping $q : [0, T] \times \mathbb{R}^2 \to \mathcal{P}([\sigma_1, \sigma_2])$.

Temptation: Define the relaxed controlled state process by

$$X_t = \int_0^t \left(\int_A a q(s, da) \right) dW_s$$

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However

$$\lim_n \langle \alpha^n \cdot W, \alpha^n \cdot W \rangle_T = \lim_n T \sum_{k=1}^{n} \frac{\xi_k^2}{n} = \left(\int a^2 \mu(da) \right) T \qquad (LLN)$$

$$\neq \left(\int a \mu(da) \right)^2 T$$

$$= \langle X, X \rangle_T$$

State dynamics in terms of a martingale problem

- (X_t), (Y_t) canonical processes
- $q_1, q_2 : [0, T] \times \mathbb{R}^2 \to \mathcal{P}([\sigma_1, \sigma_2])$ **'relaxed controls'**
- P^{q_1,q_2} is a feasible distribution if X and Y are martingales and

$$d\langle X, X\rangle_t = \int a^2 q_1(X_t, Y_t, da)dt$$

$$d\langle Y, Y\rangle_t = \int b^2 q_2(X_t, Y_t, db)dt$$

$$d\langle X, Y\rangle_t = \int \int \rho ab q_1(X_t, Y_t, da) q_2(X_t, Y_t, db)dt$$

Equilibria in relaxed controls

Theorem

Let $\rho > \sqrt{\frac{\sigma_1+\sigma_2}{2\sigma_2}}$. Then the game has a value in relaxed controls (given in closed form) and the tuple $(q_1^*, q_2^*) \in \mathcal{V} \times \mathcal{V}$ defined by

$$q_1^*(x,y) = \begin{cases} \frac{1}{\sigma_2-\sigma_1}\left(\left(\sigma_2 - \frac{\sigma_1+\sigma_2}{2\rho^2}\right)\delta_{\sigma_1} + \left(\frac{\sigma_1+\sigma_2}{2\rho^2} - \sigma_1\right)\delta_{\sigma_2}\right), & \text{if } x \le y, \\ \delta_{\frac{\sigma_1+\sigma_2}{2\rho}}, & \text{if } x > y, \end{cases}$$

$$q_2^*(x,y) = \begin{cases} \delta_{\frac{\sigma_1+\sigma_2}{2\rho}}, & \text{if } x \le y, \\ \frac{1}{\sigma_2-\sigma_1}\left(\left(\sigma_2 - \frac{\sigma_1+\sigma_2}{2\rho^2}\right)\delta_{\sigma_1} + \left(\frac{\sigma_1+\sigma_2}{2\rho^2} - \sigma_1\right)\delta_{\sigma_2}\right), & \text{if } x > y, \end{cases}$$

is a saddle point.

SDE representation of relaxed controlled states

$q_1, q_2 : \mathbb{R}^2 \to \mathcal{P}([\sigma_1, \sigma_2])$ **'relaxed controls'**

Then the states solve

$$dX_t = \int_{\sigma_1}^{\sigma_2} a\, q_1(t, X_t, Y_t)(da)\, dW_t^1 + \sqrt{\mathrm{Var}(q_1(t, X_t, Y_t))}\, d\tilde{B}_t^1$$

$$dY_t = \int_{\sigma_1}^{\sigma_2} b\, q_2(t, X_t, Y_t)(db)\, dW_t^2 + \sqrt{\mathrm{Var}(q_2(t, X_t, Y_t))}\, d\tilde{B}_t^2,$$

$\tilde{B}^1, \tilde{B}^2$ new independent BMs

Conclusion

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Literature

- S. Ankirchner, N. Kazi-Tani and J. Wendt. *The role of correlation in diffusion control games*. SIAM Journal of Control and Optimization. 2024.

Thank you!