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Random Dynamical Systems[1]

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[1]This file still contains type mistakes. It will be updated during the workshop. October 16, 2020

1. Introduction

The theory of dynamical systems deals with the qualitative long time behavior of (autonomous) differential/difference equations. Consider the initial value problem in $\mathbb{R}^d$

$$u' = f(u), \quad u(0) = u_0.$$

Suppose that f is regular enough such that the initial value problem has a unique solution on $\mathbb{R}^+$. Then we can study the solution mapping

$$(t, u_0) \mapsto u_{u_0}(t)$$

General properties of this mapping give us an object called a dynamical system

DEFINITION 1. *Let H be a Polish (metric complete separable) space. Consider a family of mappings*

$$(\varphi(t, \cdot))_{t\in\mathbb{R}^+}, \quad \varphi(t, \cdot) : H \to H$$

satisfying the semigroup property:

$$\begin{aligned} \varphi(t, \cdot) \circ \varphi(\tau, \cdot) =& \varphi(t + \tau, \cdot), \quad \text{for all } t,\, \tau \geq 0 \\ \varphi(0, \cdot) =& \mathrm{id}_H, \end{aligned}$$

then φ is called a dynamical system. The dynamical system is called continuous if

$$\varphi(t, \cdot) : H \to H$$

is continuous for any $t \in \mathbb{R}^+$.

For instance the following objects are of interest in the theory of dynamical systems:

- A state $u_s \in H$ is called stationary point if

$$\varphi(t, u_s) = u_s \quad \text{for all } t \geq 0.$$

- Suppose the dynamical system φ has the stationary point $u_s \in H$. We call φ *stable* with respect to u_s if for any $\epsilon > 0$ we have a $\delta > 0$ such that

$$\sup_{t\geq 0} d_H(\varphi(t, u_0), u_s) < \epsilon \quad \text{if } d_H(u_0, u_s) < \delta.$$

- u_s is called attracting for φ if there exists a neighborhood U of u_s such that for $u_0 \in U$ we have

$$\lim_{t\to\infty} \varphi(t, u_0) = u_s.$$

  A dynamical system which is both stable and attracting is called asymptotically stable.
- Suppose from now on that H is a Banach space and for simplicity that the stationary point $u_s = 0$. We are looking for the sets u_0 of states in H such that

$$\lim_{t\to\infty} \|\varphi(t, u_0)\| = 0.$$

  This set formed by these states is called the stable set M^-. Sometimes M^- has the structure of a C^k or Lipschitz manifold. For this we assume that the linearization of the dynamical system has an invariant linear subspace H^- of H and that along this subspace the linearization tends to zero for $t \to \infty$. Then, under particular conditions, M^- can be represented by a regular manifold:

$$M^- = \{(m^-(u^-), u^-) : u^- \in H^-\}$$

and $m^- : H^- \to H \ominus H^-$ is a C^k or Lipschitz mapping.

- Consider a family of dynamical systems φ^λ where $\lambda \in \Lambda \in \mathbb{R}$. An element $\lambda_0 \in \Lambda$ is called bifurcation point if the structure of the set of stationary points is different for $\lambda \in (-\infty, \lambda_0) \cap U_{\lambda_0}$ and $\lambda \in (\lambda_0, +\infty) \cap U_{\lambda_0}$ where and U_{λ_0} is a neighborhood of λ_0.
- The prototype of a set in the state space H describing the long time behavior of the dynamical system φ is the (global) attractor. A dynamical system φ has a global attractor A if A is a compact and nonempty set in H. In addition A is invariant for φ:

$$\varphi(t, A) = A, \quad \text{for } t \geq 0.$$

Moreover A attracts all bounded sets $D \subset H$:

$$\lim_{t\to\infty} d_H(\varphi(t, D), A) = 0, \quad d_H(A, B) = \sup_{a\in A} \inf_{b\in B} d_H(a, b)$$

and A, B are nonempty sets.

Attractors allow further investigations. For instance, one is interested to estimate the Hausdorff - or fractal dimension of an attractor. Another interesting question is when an attractor attracts states exponentially fast or what is the fixed point structure inside the attractor.

In the following we will consider some of these objects under random perturbations. In particular, we will introduce the term noise as a random perturbation. We will introduce, random attractor, random invariant manifolds and random fixed points for noisy dynamical systems. For this we need some base from the probability theory.

2. Stochastic basics

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(E, \mathcal{E})$ be some measurable space where E is a Polish space and $\mathcal{E}$ is the Borel-σ-algebra. A measurable mapping

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$$

is called a random variable. Let $\mathbb{T}$ be the set $\mathbb{R}^+$ or $\mathbb{R}$ or a interval equipped with the usual metric. A family of random variables $(X(t))_{t\in\mathbb{T}}$, $X(t) \in E$ is called a random process. The mappings

$$\mathbb{T} \ni t \mapsto X(t, \omega), \quad \omega \in \Omega$$

are called the paths or the trajectories of a random process. We call the random process X' a version of X if

$$\mathbb{P}(\{\omega \in \Omega : X(t, \omega) = X'(t, \omega)\}) \equiv \mathbb{P}(X(t) = X'(t)) = 1 \quad \text{for all } t \in \mathbb{T}.$$

Two random processes are called indistinguishable if

$$\mathbb{P}(\{\omega \in \Omega : X(t, \omega) = X'(t, \omega) \quad \text{for all } t \in \mathbb{T}\}) = 1$$

where we assume that the set inside the probability is measurable. The existence of random processes can be proved by the Kolmogorov fundamental theorem, [Bau96] Chapter III. This construction gives us random processes with very irregular trajectories. For further applications we are rather interested in random processes with continuous or Hölder continuous paths. A process with $T = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}^+$ is called

β Hölder continuous ($\beta \in (0,1]$) if the restriction to any compact $[a,b]$ interval of $\mathbb{T}$ has a finite Hölder semi norm:

$$\sup_{a\le s<t\le b}\frac{\|X(t)-X(s)\|}{|t-s|^{\beta}}<\infty.$$

In particular, we are looking for a version of a process which is regular in this sense. Sufficient conditions are formulated in the Kolmogorov regularity theorem, see Bauer [**Bau96**] Section 39 or Kunita [**Kun90**] Theorem 1.4.1:

THEOREM 2. *Let X be a random process with values in the (separable?) Banach space E. Suppose that*

$$\mathbb{E}\|X(t)-X(t')\|^{\gamma}\le C|t-t'|^{1+\epsilon}$$

for all t, $t'\in\mathbb{T}$ for positive constants C, γ, ϵ. Then X has a continuous version X'. In addition for a $0<\beta<\epsilon/\gamma$ almost every path of X' is β locally Hölder continuous.

From the fundamental theorem of Kolmogorov we know that the distribution (on an appropriate σ-algebra) is given by the so called finite dimensional distributions. In particular, knowing the distribution of the random vectors $(X(t_1),\cdots,X(t_n))$ for any n tuple $(t_1,\cdots,t_n)$ of $\mathbb{T}$ the distribution of X is well determined. Suppose that for any n tuple $(X(t_1),\cdots,X(t_n))$ is a Gauß random vector. Let X by a random process. This process is called Gauß process if the vectors mentioned are Gauß vectors. To see if such a vector is a Gauß vector one has to consider the characteristic functional of this vector which has a particular structure. Suppose $E=\mathbb{R}^d$ then the density of this vector has the form determined by a covariance matrix and a mean vector. This is a general property of Gauß processes: The distribution of such a process is determined by the mean vector of all the random variables and the covariance of each pair $X(t_i)$ and $X(t_j)$ for $1\le i\le j\le n$. In particular, there exists a *covariance function* $\Gamma(s,t)$ such that

$$\Gamma(t_i,t_j)=\operatorname{cov}(X(t_i,t_j).$$

A typical Gauß process is the Brownian motion. Let E be a separable Hilbert space. A Brownian motion is a Gauß process with mean zero and covariances

$$\mathbb{E}(W(s)\otimes W(t))=\frac{1}{2}Q(|t|+|s|-|t-s|)\quad s,\,t\in\mathbb{R}.$$

Here Q is the so called covariance operator which is a symmetric non negative linear operator E of finite trace. Such a Brownian motion is called twosided because we have $\mathbb{T}=\mathbb{R}$. For Brownian motions of $\mathbb{R}^+$ we then have

$$\mathbb{E}(W(s)\otimes W(t))=Q\min(t,s)\quad s,\,t\in\mathbb{R}^+.$$

From these formulas we obtain

$$\mathbb{E}\|W(t)-W(s)\|^2\le c_2|t-s|.$$

In particular we have $W(0)=0$ almost surely. By the Gauß character of this process we are able to estimate higher even moments of $\|W(t)-W(s)\|$ by

$$\mathbb{E}\|W(t)-W(s)\|^{2n}\le c_{2n}|t-s|^n.$$

Now we can apply the Kolmogorov regularity theorem to see that W has a continuous version with Hölder continuous path for every $\beta<1/2$. In the following we will always assume that W has continuous paths.

Lemma 3. *A Brownian motion has the following properties:*

- *For $t_1 < t_2 \leq t_3 < t_4$ the random variables*
$$W(t_1) - W(t_2) \quad \text{and} \;\; W(t_3) - W(t_4)$$
are independent.
- *$W(t) - W(s)$ is a Gauß random variable with mean zero and covariance $Q|t-s|$.*

A random process is called canonic if the path space coincides with Ω, hence:

$$X(t,\omega) = \omega(t)$$

More precisely for the Brownian motion we can find the canonical version

$$(C_0(\mathbb{R},E), \mathcal{B}(C_0(\mathbb{R},E)), \mathbb{P}) \tag{1}$$

where $C_0 := C_0(\mathbb{R},E)$ is the space of continuous functions on $\mathbb{R}$ equipped with the Fréchet metric and $\mathcal{B}(C_0(\mathbb{R},E))$ is the σ-algebra generated by this metric. $\mathbb{P}$ is the Gauß measure with covariance Q and mean zero. The existence of such a process follows by the existence of a continuous version of this process. For details we refer to Bauer [**Bau96**] Section 38 and in particular Theorem 38.6.

Sometimes it will be necessary to consider a filtration for the Brownian motion. We here for simplicity consider the natural filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$

$$\mathcal{F}_t = \sigma\{\omega(s) : s \leq t\}.$$

We call a random process X adapted with respect to some filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ if $X(t)$ is $\mathcal{F}_t$ measurable.

Definition 4. *A mapping*

$$G : \Omega \times H \to H$$

is called a Carathéodory mapping if

$$x \mapsto G(\omega, x)$$

is continuous for $\omega \in \Omega$ and

$$\omega \mapsto G(x, \omega)$$

is $\mathcal{F}$ measurable.

For these mappings we have:

Lemma 5. *A Carathéodory mapping is $\mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H)$ measurable.*

3. Random dynamical systems

We introduce a model for a general noise.

Definition 6. *A quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and θ is a measurable flow:*

$$\theta : \mathbb{R} \times \Omega \mapsto \Omega$$

such that

$$\theta_t \circ \theta_\tau = \theta_t\theta_\tau = \theta_{t+\tau}, \quad \theta_0 = \mathrm{id}_\Omega$$

is a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F}$ measurable mapping. In addition we have that $\theta_t\mathbb{P} = \mathbb{P}$ for $t \in \mathbb{R}$.

For the partial mapping $\theta(t,\cdot)$ we have written θ_t. These operators describe the dynamics of a random perturbation which is called noise. In particular the probability measure $\mathbb{P}$ is persistent with respect to the action of the *shifts* θ_t. In this sense we have that noise is stationary noise.

We have the following examples for a metric dynamical system

- We consider the canonical Brownian motion introduced in the last section. In addition we introduce the shift operator

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}.$$

Then

$$\theta : \mathbb{R} \times C_0(\mathbb{R}, E) \to C_0(\mathbb{R}; E)$$

is well defined. For the measurability of this mapping which is based on Lemma 43 we refer to Arnold [Arn98] Appendix. In particular we see that by Lemma 3

$$\omega(t+s) - \omega(t)$$

have a distribution which is independent of t. More precisely, the random process $s \mapsto \theta_t\omega(s)$ has the same distribution as the canonical process $s \mapsto \omega(s)$ such that we obtain the stationarity of $\mathbb{P}$ with respect to the shifts θ_t. Hence we have the metric dynamical system

$$(C_0(\mathbb{R}, E), \mathcal{B}(C_0(\mathbb{R}, E)), \mathbb{P}, \theta).$$

- We consider the canonical stationary Ornstein Uhlenbeck process (for a while) on $\mathbb{R}$

$$(C(\mathbb{R}, \mathbb{R}), \mathcal{B}(C(\mathbb{R}, \mathbb{R})), \mathbb{P}_Z, \tilde{\theta})$$

where the shifts are given by $\tilde{\theta}_t z = z(\cdot + t)$ for $z \in C(\mathbb{R}, \mathbb{R})$. The measure $\mathbb{P}_Z$ is the distribution on $\mathcal{B}(C(\mathbb{R}, \mathbb{R}))$ of the random process

$$t \mapsto Z(\theta_t\omega).$$

This stationary process is generated by the random variable

$$Z(\omega) = \int_{-\infty}^{0} e^r \omega(r) dr$$

which is defined on a set Ω' of $\omega \in C_0(\mathbb{R}, \mathbb{R})$ (see the last example) of full measure which is θ-invariant:

$$\theta_t\Omega' = \Omega' \quad \text{for all } t \in \mathbb{R}.$$

A similar construction is possible in a separable Hilbert space E.

- A generalization of the first example is the canonical fractional Brownian

$$(C_0(\mathbb{R}, E), \mathcal{B}(C_0(\mathbb{R}, E)), \mathbb{P}_H, \theta).$$

In particular $\mathbb{P}_H$ is the distribution of a special Gauß process with a particular covariance depending on a parameter $H \in (0, 1)$ which is called the Hurst parameter. For $H = 1/2$ we obtain as a special case the Brownian motion.

Definition 7. *A metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called ergodic if all elements $A \in \mathcal{F}$ having the property*

$$\mathbb{P}(\theta_t A \Delta A) = 0 \quad \text{for all } t \in \mathbb{T}$$

have $\mathbb{P}(A) \in \{0,1\}$. In particular, for all θ invariant sets $A \in \mathcal{F}$ we have $\mathbb{P}(A) \in \{0,1\}$.

Ergodicity of a metric dynamical system has some consequences. In particular this special Birkhoff ergodic theorem holds:

Theorem 8. *We consider $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ a metric dynamical system which is ergodic. In addition suppose that $f : \Omega \to \bar{\mathbb{R}}$ is an integrable random variable. Then there exists a θ invariant set $\Omega' \in \mathcal{F}$ of $\mathbb{P}$ measure one such that*

$$\lim_{t\to+\pm\infty} \frac{1}{t}\int_0^t f(\theta_r\omega)dr = \mathbb{E}f \quad \text{for } \omega \in \Omega'.$$

We introduce a special class of random variables.

Definition 9. *Given a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. A nonnegative random variable X is called tempered (from above) if*

$$\lim_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} = 0 \quad \text{almost surely}$$

A positive random variable X is called tempered from below if X^{-1} is tempered from above.

We note if there exists a set $\Omega' \in \mathcal{F}$ of full measure where this convergence, then Ω' is θ invariant. A sufficient condition for temperedness is given by

Lemma 10. *Suppose that for some $a < b$*

$$\mathbb{E} \sup_{t\in[a,b]} \log^+ X(\theta_t\omega) < \infty$$

then X is tempered.

Temperedness describes the property that the stationary random process $X(\theta_t\omega)$ is subexponentially growing or in other words that the random process $\log^+ X(\theta_t\omega)$ grows sublinearly. Processes with this property are investigated in Arnold [Arn98] Proposition 4.1.3. From these considerations we can derive that the only alternative of tempered is that

$$\limsup_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} = \infty. \tag{2}$$

In particular the limits for $t \to +\infty$ and $t \to -\infty$ are the same. Recall that the metric dynamical system is ergodic. Then we have as a limit superior 0 with probability one or ∞ with probability one. In particular,if we can show that

$$\limsup_{t\to+\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} < \infty \quad \text{or} \ \limsup_{t\to-\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} < \infty$$

with probability larger the zero then X is tempered.

Definition 11. *A mapping*

$$\varphi : \mathbb{R}^+ \times \Omega \times H \to H$$

which is $\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}\otimes\mathcal{B}(X), \mathcal{B}(X)$ *measurable is called a random dynamical system (RDS) if the cocycle property holds:*

$$\varphi(t,\theta_\tau\omega,\cdot)\circ\varphi(\tau,\omega,\cdot)=\varphi(t+\tau,\omega,\cdot)\quad \text{for all } t,\,\tau\in\mathbb{R}_+,\,\omega\in\Omega,$$
$$\varphi(0,\omega,\cdot)=\mathrm{id}_H\quad \text{for all } \omega\in\Omega$$

If the mapping $x\mapsto\varphi(t,\omega,x)$ *is continuous for all* $\omega\in\Omega$ *and* $t\geq 0$ *then we call the RDS continuous.*

The cocycle property generalized the semigroup property of a dynamical system. This can be seen if we delete the ω's in the above formula. We also note that this property has to be satisfied for all $\omega\in\Omega$. If this property only holds almost surely where the exceptional sets for these formulas depend say on τ then we call this property the *crude cocycle property*. If a mapping φ only has this property it is possible in many cases to find a version of φ which is an RDS. We refer to Arnold and Scheutzow [**AS95**]. Suppose that the above property is only satisfied for a set Ω' which is θ invariant. Then we can find a version of φ which has the cocycle property for all $\omega\in\Omega$ by changing or defining $\varphi(t,\omega,\cdot)=\mathrm{id}_H$ on Ω'^c.

RDS are generated by random differential equations

$$u'=f(\theta_t\omega,u),\quad u(0)=u_0\in H$$

where f is a Carathéodory mapping from $\Omega\times H$ to H sufficiently regular such that the above differential equation has a unique global solution say in $C(\mathbb{R}^+,H)$ which depends measurably on (ω,u_0). Then we can define

$$\varphi(t,\omega,u_0)=u_{\omega,u_0}(t)$$

where the latter term is the solution for a noise path ω and an initial state $u_0\in H$.

Consider the Ito equation in $\mathbb{R}^d$

$$du=f(u)dt+g(u)d\omega,\quad u(0)=u_0\in\mathbb{R}^d.$$

where f, g are Lipschitz mappings

$$f:\mathbb{R}^d\to\mathbb{R}^d,\quad g:\mathbb{R}^d\to L(\mathbb{R}^m,\mathbb{R}^d).$$

ω describes a standard Brownian motion in $\mathbb{R}^m$. Then for any u_0 there exists a continuous random process solving this Ito equation. This process is unique modulo $\mathbb{P}$. It is not trivial to construct for the solution of a differential equations an RDS. However one can show that the crude cocycle property holds. Hence techniques from Arnold and Scheutzow [**AS95**] allow to construct an RDS version of the solutions of the Ito equations.

Consider a stochastic partial stochastic differential equation (in the Ito sense)

$$\partial_t u=(\Delta u+f(u))dt+g(u)d\omega$$

or a stochastic evolution equation in some infinite dimensional Hilbert space. It is a delicate question to prove the cocycle property for spde where the diffusion part (the operator g) is non trivial. In the next section we will study a simple version of theses equations.

4. Random dynamical systems and stochastic evolution equations

In this section we would like to generate an RDS by a stochastic evolution equation on a separable Hilbert space H:

$$du + Au\,dt = F(u)\,dt + dW, \quad u(0) = u_0 \in H, \tag{3}$$

where A is a linear *unbounded* operator which will be introduced below, F is assumed (for simplicity) to be a non linear operator which is Lipschitz continuous:

$$\|F(u) - F(v)\| \le L\|u - v\|, \quad \text{for } u,\, v \in H$$

and W is a Brownian motion where the distribution is generated by the trace class covariance Q. The initial condition u_0 can be chosen as an $\mathcal{F}_0$ measurable random variable.

We now describe the operator A. In the following A should be generated by differential operators like the Laplace operator $-\Delta$ defined on a bounded domain with smooth boundary satisfying homogeneous Neumann[1] or Dirichlet boundary conditions. Or more general we could consider differential operators of the type

$$\sum_{|j|,|k|\le i} (-1)^{|j|} D^j(a_{jk}(x) D^k u(x))$$

where the coefficients a_{jk} are sufficiently regular and positive definite in some sense, see Zeidler [Zei90] Definition 22.42.
Let A be a (unbounded) linear operator in the separable Hilbert space H defined on $D(A)$ such that $D(A) = H$. We assume that A is positive and symmetric

$$(Au, u) \ge c\|u\|^2, \quad (Au, v) = (Av, u) \quad u,\, v \in D(A)$$

for a positive constant c. In addition, we assume that A has a compact inverse. These conditions ensure that A^{-1} satisfies the assumptions of the main spectral theorem of compact operators in Hilbert spaces. In particular, the set of eigenvectors of A^{-1} form a complete orthonormal system $(e_i)_{i\in\mathbb{N}}$ in H and the related eigenvalues are positive and only have the cluster point zero. Then A has the spectrum

$$0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$$

and the associated eigenvectors are e_i. All the λ_i have finite multiplicity. The elements $u \in H$ can be presented by the orthonormal base $(e_i)_{i\in\mathbb{N}}$, see [AG77]. In particular,

$$u = \sum_{i=1}^\infty \hat{u}_i e_i, \quad \|u\|^2 = \sum_{i=1}^\infty \hat{u}_i^2$$

and

$$D(A) = \{u \in H : \sum_{i=1}^\infty \lambda_i^2 \hat{u}_i^2 =: \|u\|_1^2 = \|Au\|^2 < \infty\}.$$

[1]In the case of Neumann boundary conditions we should consider $-\Delta + a$ where a is a positive constant.

By the spectral theorem we can construct for $\alpha > 0$ the operators A^α and $e^{-At} =: S(t)$:

$$A^\alpha u = \sum_{i=1}^\infty \lambda_i^\alpha \hat{u}^i e_i, \quad u \in D(A^\alpha) = \{u \in H : \sum_{i=1}^\infty \hat{u}_i^2 \lambda_i^{2\alpha} =: \|u\|_\alpha^2 < \infty\}$$

$$S(t)u = \sum_{i=1}^\infty e^{-\lambda_i t} \hat{u}_i e_i$$

where $S(t)$ can be defined on $D(A^\alpha)$ for any $\alpha \geq 0$, see [SY02] Page 67. We also note that $D(A^0) = H$. We see that the family of linear $(S(t))_{t\geq 0}$ form an operator semigroup on H:

$$S(t)S(\tau) = S(t+\tau) \quad \text{for } t,\, \tau > 0, \qquad S(0) = \mathrm{id}_H.$$

Without proof we notice some properties for S.

LEMMA 12. *The mapping $t \mapsto S(t)u$ is continuous in particular for $t = 0$ for every $u \in H$. S has the strong derivative A in $t = 0$:*

$$\lim_{t\to 0^+} \frac{S(t)u - u}{t} = -Au, \quad \textit{for all } u \in D(A). \tag{4}$$

$$\|A^\alpha S(t)\|_{L(H,H)} = \|S(t)\|_{L(H,D(A^\alpha))} \leq \frac{c_\alpha}{t^\alpha}\|u\| \quad \textit{for } t > 0,\ u \in H,\ \alpha > 0.$$

$$\|(S(t) - \mathrm{id})u\| \leq c'_\alpha t^\alpha \|A^\alpha u\| \quad \textit{for } t > 0,\ u \in D(A^\alpha),\ \alpha \in (0,1].$$

The first property describes a so called C_0 semigroup. The latter two properties are typical for so called analytic semigroups, see Pazy [Paz83] Theorem 2.6.13.

Consider the following operator differential equation on H:

$$\frac{du}{du} + Au = 0, \quad u(0) = u_0 \in H$$

Then it is easily checked by considering the modes $\hat{u}_i(t)$ of $u(t)$ that the solution of differential equation is given by

$$u(t) = S(t)u_0.$$

That is why A is called the generator of the semigroup S. However this is true for general generators of C_0 semigroups.

A typical example is the following. Let H be the space $L_2(D)$ of square integrable functions. Let especially D be a bounded smooth domain in $\mathbb{R}^d$. Then we can introduce the Sobolev spaces $W_2^i(D)$ of all functions over D which are in $L_2(D)$ having generalized derivatives (in the distributional sense) which can be presented by functions in $L_2(D)$. These spaces are Hilbert spaces with the norm

$$\|u\|_{W_2^1}^2 = \sum_{j:|j|=1}^{i} \|D^j u\|_{L_2}^2.$$

Here j is the multi index $(j_1, \cdots, j_d)$ such that $j_1 + \cdots + j_d = |j|$, j_k is the order of differentiation with respect to the k coordinate. For $j_k = 0$ we set $D_k^0 u = u$. We note that is makes sense to define for $W_2^1(D)$ a boundary operator allowing to define *in some sense* the function on the boundary. Define how

$$W_2^{1,0}(D) = \overline{C_0^\infty(D)}^{W_2^1}.$$

Then the boundary value of these elements from this space is zero. That is why we can set

$$A = -\Delta = -\sum_{i=1,\cdots,d} D_i^2$$

with domain $D(A) = W_2^2(D) \cap W_2^{1,0}(D)$. Functions from this space satisfy zero boundary conditions. For functions from this space $-\Delta$ is a positive symmetric operator. By the compact embedding $D(A) \subset H$ we know that $-\Delta$ has a positive spectrum λ_i where the associated eigenvectors generate a complete orthonormal system in H. So $-\Delta$ has all the properties we have mentioned above. In particular (3) can be expressed by

$$\partial_t u = \Delta u, \quad u(0) = u_0 \in L_2(D), \quad u|_{\partial D} = 0.$$

Formally the solution could be expressed by $u(t) = S(t)u_0$. By Lemma 12 for $t > 0$ $u(t) \in D(A)$ such that the homogenous Dirichlet boundary conditions are fulfilled automatically.

The most simple semigroup $S(t)$ is give on $H = \mathbb{R}$ by $S(t) = e^{-\lambda t}$ with generator $A = \lambda$. For $H = \mathbb{R}^d$ the semigroups are defined by the matrix exponentials $S(t) = e^{-At}$ where A is a (d, d) matrix. Consider for instance the one dimensional differential equation

$$\frac{du}{dt} + \lambda u = F(t, u), \quad u(0) = u_0$$

then according to the variation of constants formula we have the following equivalent integral equation to determine the solution u

$$u(t) = e^{-\lambda t}u_0 + \int_0^t e^{-\lambda(t-r)} f(r, u(r))dr.$$

In particular, a solution u solves the above differential equation if and only if the latter integral equation is satisfied. However consider the integral equation we avoid to differentiate functions. In the following we consider the nonautonomous evolution equation

$$\frac{dv}{dt} + Av = F(t, v), \quad v(0) = v_0 \in H \tag{5}$$

or the stochastic evolution equation (3) where A is the generator of a C_0 semigroup S and F is a sufficiently regular function. If $F(t, u)$ is given by $F(\theta_t\omega, u)$ then we call (3) a random evolution equation. We assume that $(\omega, u) \mapsto F(\omega, u)$ is a Carathéodory map.

Definition 13. *A continuous function in H is called mild solution to (3) if*

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(r, u(r))dr$$

holds for $t \in [0, T]$ of $t \geq 0$. A continuous random $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process is called mild solution to (3) if u satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(u(r))dr + \int_0^t S(t-r)dW(r) \quad \textit{for all } t \geq 0 a.s. \tag{6}$$

The latter integral in the last equation is the Ito integral. The Ito integral has some disadvantages in the sense of dynamical system. The Ito integral in general is defined as a limit in probability of a sequence of approximate Ito integrals. This means that this integral is only defined almost surely where the exceptional sets may depend on the integrand. In the Ito integral appearing in (6) we could avoid this because the integrand does not depend on ω. But to find a proper expression for $\|u(t)\|^2$ we had to apply the chain rule which is the Ito formula for so called Ito processes. Then (3) would give us an expression for the square of the norm containing

$$2\int_0^t (u(r), dW(r))$$

where the exceptional set may depend on u_0. This dependence creates some problems to generate by (3) and hence by (5) an RDS.

We would like to apply the conjugating method to find a version of the solution of (3), (5) which is an RDS. Let us also assume that A generates an exponentially stable semigroup such that the properties of Lemma 12 hold. In particular when A is positive and symmetric then A generates an exponential stable semigroup. Then we have for the A introduced above

$$\|S(t)\|_{L(H)} \leq e^{-\lambda_1 t}$$

we note that

$$\int_0^t S(t-r)dW(r)$$

solves the Ito equation

$$dZ + AZdt = dW, \quad Z(0) = 0.$$

This kind of problem is studied intensively in DaPrato and Zabczyk [DZ14] Section 4.

LEMMA 14. *Let W be a two sided Brownian motion with a covariance of trace class in H and A be a generator with the properties from above (in particular A is the generator of an analytic semigroup). Then a version of the random process generated by the Ito integral*

$$\int_0^t S(t-r)dW(r)$$

is given by

$$S(t)W(t) + A\int_0^t S(t-r)W(r)dr. \tag{7}$$

PROOF. Let us consider a canonical Brownian motion $W(t,\omega) = \omega(t)$ on the probability space defined in (1). We apply the integration by parts technique for Ito integral with non random integrand, see Øksendal [Øks03] Chapter 3. This formula also makes sense in infinite dimensions by limit transition and by the fact that $S(t)$ has a strong derivative given by $AS(t)$. The integral

$$A\int_0^t S(t-r)\omega(r)dr$$

exists by Pazy [Paz83] Theorem 4.3.5 because $\omega(0) = 0$ and ω is β-Hölder continuous for $\beta < 1/2$. In particular, this mapping in t is β Hölder continuous. □

LEMMA 15. *The random variable with values in H given by*

$$Z(\omega) = \begin{cases} A\int_{-\infty}^{0} S(-r)\omega(r)dr & : \quad \omega \in \Omega' \\ 0 & \quad\;\; \omega \notin \Omega' \end{cases}$$

is well defined where

$$\Omega' = \{\omega \in C_0 : \sup_{r\in[0,1]} \|\theta_i\omega(r)\| \text{ has a subexponential growth for } i \to \pm\infty\}.$$

This set is θ invariant and has measure one. Then the random process

$$t \mapsto Z(\theta_t\omega)$$

solves

$$dz + Azdt = dW, \quad z(0) = Z(\omega).$$

Z is $\mathcal{F}_0$ measurable.

PROOF. Replacing in

$$S(t)\omega(t) + A\int_0^t S(t-r)\omega(r)dr$$

(see (7)) ω by $\theta_{-t}\omega$ and having the having the subexponential growth of ω and the exponential stability of S in mind we obtain that Z is well defined. □

We now introduce the conjugacy of two RDS.

DEFINITION 16. *Let φ and ψ two RDS over the same metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and suppose that there exists a Carathéodory map*

$$T : \Omega \times H \to H$$

such that

$$x \to T(\omega, x)$$

is a homeomorphism for $\omega \in \Omega$ where the inverse is denoted by $T^{-1}(\omega, \cdot)$ and that $\omega \to T^{-1}(\omega, x)$ is a Carathéordory map. We call φ and ψ conjugated if we have

$$\psi(t, \omega, v) = T(\theta_t\omega, \varphi(t, \omega, T^{-1}(\omega, v))) \quad \text{for all } v \in H,\ \omega \in \Omega,\ t \geq 0. \tag{8}$$

REMARK 17. *Defining a ψ by the right hand side of (8) gives us a continuous RDS.*

We now deal with the solution of the following random evolution equation

$$\frac{dv}{dt} + Av = F(v + Z(\theta_t\omega)), \quad v(0) = v_0 \in H. \tag{9}$$

Recall that (for simplicity) F is supposed to be Lipschitz continuous on H with Lipschitz constant L and the random variable Z is given in Lemma 15.

THEOREM 18. *The problem (9) has a unique mild solution in $C([0,T], H)$ for any $T > 0$, $\omega \in \Omega$ and $v_0 \in H$. In particular, this solution generates a continuous random process which is $(\mathcal{F}_t)_{t\in[0,T]}$ adapted. In addition, this solution generates a continuous RDS.*

PROOF. Suppose there is a mild solution on some interval $[0, S(\omega, v_0))$ then this solution is defined for any $t \in \mathbb{R}^+$. Indeed, apply the Gronwall lemma to $w(t) = \|v(t)\| e^{\lambda_1 t}$ which satisfies the following equation

$$w(t) \leq \|v_0\| + \int_0^t e^{\lambda_1 s}(L\|w(r)\| + L\|Z(\theta_r \omega)\| + c)dr.$$

This allows to conclude that

$$\|v(t)\| \leq e^{-\lambda_1 t}\|v_0\| + \int_0^t e^{-\lambda_1 (t-r)}(L\|Z(\theta_r \omega)\| + c)dr.$$

Hence, if there exists a solution then this solution will never explode. Then the solution can be extended to any interval $[0, T]$. Suppose there exist two solutions v_1, $v_2$with the same initial condition. Then we have for the norm of the difference

$$\|v_1(t) - v_2(t)\| \leq \int_0^t L\|v_1(r) - v_2(r)\|dr.$$

From the Gronwall lemma again we can conclude that this norm difference is equal to zero so that $v_1 = v_2$. To see the existence of a solution we can apply the Banach fixed point theorem: Let $\mathcal{T}_{v_0,\omega}$ be the mapping on $C([0, T], H)$

$$\mathcal{T}_{v_0,\omega}(v)[t] = S(t)v_0 + \int_0^t S(t - r)F(v(r) + Z(\theta_r \omega))dr.$$

This mapping is a contraction

$$\sup_{t \in [0,T]} \|\mathcal{T}_{v_0,\omega}(v_1)[t] - \mathcal{T}_{v_0,\omega}(v_2)[t]\| \leq LT \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|, \; v_1, \; v_2 \in C([0, T], H)$$

assuming that T chosen small enough such that $LT < 1$. Doing the same on $[T, 2T]$, $[2T, 3T]$... with initial conditions $v(T)$, $v(2T), \cdots$. Having these fixed points we can concatenate these functions to one function on any interval $[0, nT]$, see below how to construct a concatenation of a solution defined on neighbored intervals. To see the continuous dependence on the initial condition $v_0 \in H$ we refer to the parameter Banach fixed point theorem, see Zeidler, [Zei86] Chapter 1. In particular the contraction constant is independent of v_0 and

$$H \ni v_0 \to S(\cdot)v_0 + \int_0^\cdot S(\cdot - r)F(v(r) + Z(\theta_r \omega))dr = \mathcal{T}_{v_0,\omega}(v) \in C([0, T], H)$$

is continuous for every $v \in C([0, T], H)$

We define a mapping

$$\varphi(t, \omega, v_0) = v(t) = v_{v_0,\omega}(t)$$

where v is a solution to (9) defined on $\mathbb{R}$. Then we have the cocycle property by concatenation:

$$\begin{aligned}\varphi(t+\tau,\omega,v_0) =& S(t+\tau)u_0 + \int_0^{t+\tau} S(t+\tau-r)F(v(r,\omega)+Z(\theta_r\omega))dr\\ =& S(t)(S(\tau)v_0 + \int_0^{\tau} S(\tau-r)F(v(r,\omega)+Z(\theta_r\omega))dr)\\ &+ \int_{\tau}^{t+\tau} S(t+\tau-r)F(v(r,\omega)+Z(\theta_r\omega))dr.\\ =& S(t)(\varphi(\tau,\omega,v_0) + \int_0^{t} S(t+\tau-r)F(y(r+\tau,\omega)+Z(\theta_{r+t}\omega))dr\\ =& \varphi(t,\theta_\tau\omega,\varphi(\tau,\omega,v_0)).\end{aligned}$$

It remains to state the measurability of $(t,\omega,v_0)\mapsto\varphi(t,\omega,v_0)$. To see this we note that the fixed point of $\mathcal{T}_{v_0,\omega}$ can be constructed by iteration:

$$v^0\equiv v_0,\quad v^{i+1}=\mathcal{T}_{v_0,\omega}(v^i),\quad \lim_{i\to\infty}\mathcal{T}_{v_0,\omega}(v^i)=v$$

where $(t,\omega)\mapsto\mathcal{T}_{v_0,\omega}(v^i)[t]$ is measurable supposing that v^i is measurable. Straightforwardly v^0 is measurable. Since the pointwise limit of measurable functions is measurable we have that

$$v(t)_{v_0,\omega}(t)=\varphi(t,\omega,v_0)$$

is measurable. The measurability with respect to all arguments follows by Lemma 43. □

Remark 19. *It also makes sense to consider mild solutions to (9) if v_0 is a random variable with values in H. In particular, if this random variable is measurable with respect to $\mathcal{F}_0$ then the solution is still $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ adapted.*

Now we are in a position to formulate the main theorem of this section

Theorem 20. *Suppose that W is a trace class Brownian motion in H, the linear operator A and the nonlinear operator F have the above properties. Then the mild solution to (3) generates a continuous RDS on H.*

Proof. We consider the transform

$$T(\omega,x)=x+Z(\omega)$$

and the inverse transform

$$T^{-1}(\omega,x)=x-Z(\omega)$$

we define $u(t,\omega)=v(t,\omega)+Z(\theta_t\omega)$ where v is the solution of (9). Then we can apply Remark 34. In particular adding together the equation for the mild solution of v and Z we see that u is a mild solution to (6). □

5. Stationary points

A (deterministic) dynamical system φ has a stationary point $u_s\in H$ if

$$\varphi(t,u_s)=u_s\quad\text{for } t\geq 0.$$

We would like to generalize this term to RDS.

Definition 21. *A random variable $u_s \in H$ is called a (random) stationary point if*

$$\varphi(t,\omega,u_s(\omega)) = u_s(\theta_t\omega), \quad \text{for all } t \geq 0,\ \omega \in \Omega.$$

Since θ_t is a stationary shift for the probability measure $\mathbb{P}$ the random process $t \mapsto u_s(\theta_t\omega)$ is a stationary process. In particular, if the φ is generated by a random differential equation then $t \mapsto u_s(\theta_t\omega)$ gives a stationary solution.

With respect to an ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ we consider the dynamical system φ generated by the one dimensional random differential equation

$$\frac{dw}{dt} + a(\theta_t\omega)w = f(\theta_t\omega), \quad w(0) = w_0. \tag{10}$$

We assume that $a \in L_1(\Omega)$, $\mathbb{E}a > 0$ and $|f|$ is a tempered random variable. A stationary point for this RDS is given by

$$w_s(\omega) = \int_{-\infty}^0 e^{-\int_r^0 a(\theta_q\omega)dq} f(\theta_r\omega)dr.$$

The integral on the right hand side exists because by the temperedness of f the mapping $t \mapsto |f(\theta_{-t}\omega)|$ grows subexponentially for $t \to \infty$ and by the Birkhoff ergodic theorem

$$\int_r^0 a(\theta_q\omega)dq \sim -r\mathbb{E}a \quad \text{for large } |r|$$

such that the exponential term goes to zero exponentially fast.
Indeed a solution w to (10) is given by

$$w(t) = w_0 e^{-\int_0^t a(\theta_r\omega)dr} + \int_0^t e^{-\int_r^t a(\theta_r\omega)dr} f(\theta_r\omega)dr. \tag{11}$$

Replacing w_0 by $w_s(\omega)$ we see easily that w_s is a stationary point. We also see easily that for large t

$$\lim_{t\to\infty} |\varphi(t,\omega,w_0(\omega)) - w_s(\theta_t\omega)| e^{(\mathbb{E}a-\epsilon)t}$$

tends to zero. In particular $\varphi(t,\omega,w_0(\omega))$ approaches $w_s(\theta_t\omega)$ with exponential speed. Indeed we have

$$|\varphi(t,\omega,w_0(\omega)) - \varphi(t,\omega,w_s(\omega))| \leq |w_0(\omega) - w_s(\omega)| e^{-\int_0^t a(\theta_r\omega)dr}$$

By the Birkhoff ergodic theorem

$$\int_0^t a(\theta_r\omega)dr \sim t\mathbb{E}a \quad \text{for large } t$$

which shows the desired convergence. Let now w_0 be a tempered random variable (in particular $|w_0|$ is tempered). Then we have

$$\lim_{t\to\infty} |\varphi(t,\theta_{-t}\omega,w_0(\theta_{-t}\omega)) - w_s(\omega)| = 0 \quad \text{with exponential speed.}$$

Indeed, replace in (11) ω by $\theta_{-t}\omega$ we obtain

$$\begin{aligned}w(t) =&w_0(\theta_{-t}\omega)e^{-\int_0^t a(\theta_{r-t}\omega)dr}+\int_0^t e^{-\int_r^t a(\theta_{q-t}\omega)dq}f(\theta_{r-t}\omega)dr\\ =&w_0(\theta_{-t}\omega)e^{-\int_{-t}^0 a(\theta_r\omega)dr}+\int_{-t}^0 e^{-\int_{r+t}^t a(\theta_{q-t}\omega)dq}f(\theta_r\omega)dr\\ =&w_0(\theta_{-t}\omega)e^{-\int_{-t}^0 a(\theta_r\omega)dr}+\int_{-t}^0 e^{-\int_r^0 a(\theta_q\omega)dq}f(\theta_r\omega)dr.\end{aligned}$$

By the temperedness of $|w_0|$ the first term on the right hand side goes to zero. Straightforwardly the second terms tends to $w_s(\omega)$. This convergence (replacing ω by $\theta_{-t}\omega$) is called *pullback* convergence. To see that w_s is tempered we consider for a positive δ and $t<0$ the expression

$$\begin{aligned}e^{-\delta|t|}|w_s(\theta_t\omega)| =&e^{\delta t}|w_s(\theta_t\omega)|\\ &\le\int_{-\infty}^0 e^{\delta t}\exp\bigg(\mathbb{E}ar-\int_{r+t}^0(a(\theta_q\omega)-\mathbb{E}a)dq+\int_t^0(a(\theta_q\omega)\\ &-\mathbb{E}a)dq+\log^+|f(\theta_{r+t}\omega)|\bigg)dr.\end{aligned}$$

Applying the ergodic theorem we have the following bound of the last expression For $t\le T(\omega,\epsilon_0)<0$ and $\epsilon<\epsilon_0$ we have

$$e^{\delta t}\int_{-\infty}^0\exp(\mathbb{E}ar+3\epsilon|t+r|)dr.$$

If now $3\epsilon_0<\delta$ and $3\epsilon_0<\mathbb{E}a$ the this expression converges to zero. We have applied the sublinear behavior $\log^+|f(\theta_{r+t}\omega)|$. In order to obtain the desired property we apply (2).

In the following we introduce the term random set. We assume again that X is a Polish space.

Definition 22. *A mapping $\omega\mapsto D(\omega)\in\mathcal{P}\setminus\{\emptyset\}$ is called a random set if*

$$\{\omega:D(\omega)\cap B\}\in\mathcal{F}\quad\textit{for all } B\in\mathcal{B}(X).$$

A random set is called D a closed random set if $D(\omega)$ is closed for $\omega\in\Omega$.

We have the following lemma, See Castaing and Valadier [CV77] Chapter 3.

Lemma 23. *The following statements are equivalent.*

- *D is a closed random set*
- *The mapping*

$$\omega\mapsto\sup_{x\in D(\omega)}d_X(x,y)$$

  is a random variable for every $y\in X$.
- *There exists a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ such that*

$$\overline{\bigcup_{n\in\mathbb{N}}\{X_n\}(\omega)}^X=D(\omega).$$

In particular a closed random set contains a random variable.

We consider the RDS φ on X. A closed random set B is called forward invariant if

$$\varphi(t,\omega,B(\omega))\subset B(\theta_t\omega)\quad \text{for all } t\ge 0,\,\omega\in\Omega.$$

B is called invariant if

$$\varphi(t,\omega,B(\omega))= B(\theta_t\omega)\quad \text{for all } t\ge 0,\,\omega\in\Omega.$$

Let $\mathcal{D}$ be the set of tempered random sets such that for $D\in\mathcal{D}$ we have that

$$\omega\mapsto \sup_{x\in D(\omega)} d_X(0,x)$$

is a tempered random variable. By Lemma 23 this expression defines a random variable. A closed random set is called pullback absorbing if for every ω and every $D\in\mathcal{D}$ there exists a $T=T(\omega,D)$ such that

$$\varphi(t,\theta_{-t}\omega,D(\theta_{-t}\omega))\subset B(\omega)$$

for $t\ge T(\omega,D)$. Such a B is called pullback attracting if

$$\lim_{t\to\infty} d_X(\overline{\varphi(t,\theta_{-t}\omega,D(\theta_{-t}\omega))}^X,B(\omega))=0\quad \text{for all } \omega\in\Omega,\,D\in\mathcal{D}.$$

Here $d_X(Q_1,Q_2)$ is the so called *semi Hausdorff distance*

$$d_X(Q_1,Q_2)=\sup_{q_1\in Q_1}\inf_{q_2\in Q_2} d_X(q_1,q_2).$$

Note that for closed Q_2 we have from $d_X(Q_1,Q_2)=0$ that $Q_1\subset Q_2$.

Now we can formulate a random version of the Banach fixed point theorem which allows us to find stationary points for RDS.

THEOREM 24. *Let $(X,\|\cdot\|)$ be a Banach space. Suppose that $B\in\mathcal{D}$ is forward invariant. In addition suppose that*

$$\sup_{x\neq y\in B(\omega)}\log\frac{\|\varphi(t,\omega,x)-\varphi(t,\omega,y)\|}{\|x-y\|}\le\int_0^t k(\theta_r\omega)dr$$

and $\mathbb{E}k<0$. Then φ has a unique stationary point u_s such that $u_s(\omega)\in B(\omega)$ which is exponentially attracting and pullback exponentially attracting

$$\lim_{t\to\infty}\|\varphi(t,\omega,x(\omega))-u_s(\theta_t\omega)\|=0\quad \textit{with exponential speed,}$$
$$\lim_{t\to\infty}\varphi(t,\theta_{-t}\omega,x(\theta_{-t}\omega))=u_s(\omega)\quad \textit{with exponential speed}$$

for $x\in B$ on θ invariant set of full measure.

PROOF. Let x be a random variable in B, see Lemma 23. We would like to prove that $(\varphi(n,\theta_{-n}\omega,x(\theta_{-n}\omega)))_{n\in\mathbb{N}}$ is a Cauchy sequence in X. The limit is denoted by u_s. We have

$$\begin{aligned}
&\|\varphi(n,\theta_{-n}\omega,x(\theta_{-n}\omega))-\varphi(n+1,\theta_{-n-1}\omega,x(\theta_{-n-1}\omega))\|\\
=&\|\varphi(1,\theta_{-1}\omega,\varphi(n-1,\theta_{-n}\omega,x(\theta_{-n}\omega)))-\varphi(1,\theta_{-1}\omega,\varphi(n,\theta_{-n-1}\omega,x(\theta_{-n-1}\omega)))\|\\
\le& e^{\int_0^1 k(\theta_{-1+r}\omega)dr}\|\varphi(n-1,\theta_{-n}\omega,x(\theta_{-n}\omega))-\varphi(n,\theta_{-n-1}\omega,x(\theta_{-n-1}\omega))\|\\
\le&\exp\left(\int_0^n k(\theta_{-n+r}\omega)dr\right)\|x(\theta_{-n}\omega)-\varphi(1,\theta_{-1}\omega,x(\theta_{-n-1}\omega))\|
\end{aligned}$$

By the Birkhoff ergodic theorem we have

$$\int_0^n k(\theta_{-n+r}\omega)dr = \int_{-n}^0 k(\theta_r\omega)dr \sim \mathbb{E}k\, n \to -\infty.$$

In addition

$$\|x(\omega) - \varphi(1,\omega,x(\theta_{-1}\omega))\| \le 2 \sup_{y\in B(\omega)} \|y\|$$

where the right hand side is a tempered random variable. By the exponential character of the estimates we have found the sequence given above is a Cauchy sequence which has the limit $u_s(\omega)$. In particular, by the pointwise convergence this function in ω is a random variable. Similar we obtain for any positive sequence $(t_n)_{n\in\mathbb{N}}$ converging to ∞ the convergence to $u_s(\omega)$. Indeed we have the estimate

$$\begin{aligned}
&\|\varphi(t,\theta_{-t}\omega,x(\theta_{-t}\omega)) - \varphi([t],\theta_{-[t]}\omega,x(\theta_{-[t]}\omega))\|\\
&\le \exp\left(\int_0^{[t]} k(\theta_{r-[t]}\omega)dr\right)\|x(\theta_{-[t]}\omega) - \varphi(t-[t],\theta_{-t}\omega,x(\theta_{-t}\omega))\|
\end{aligned}$$

which converges. $[t]$ denotes the integer part of t. It is now easily seen that the limit is independent of the choice of $x \in B$. To see the invariance we note that

$$\varphi(t,\omega,u_s(\omega)) = \lim_{n\to\infty} \varphi(n,\theta_{-n+t}\omega,\varphi(t,\theta_{-n}\omega,x(\theta_{-n}\omega))).$$

We set $y(\omega) = \varphi(t,\theta_{-t}\omega,x(\theta_{-t}\omega))$ which is a element in $B(\omega)$. Then the above limit is

$$u_s(\theta_t\omega) = \lim_{n\to\infty} \varphi(n,\theta_{-n+t}\omega,y(\theta_{-n+t}\omega))$$

and hence

$$\|\varphi(n,\theta_{-n+t}\omega,x(\theta_{-n+t}\omega)) - \varphi(t,\omega,u_s(\omega))\| \le e^{\int_0^n k(\theta_{r-n+t}\omega)dr} 2 \sup_{y\in B(\theta_{-n+t}\omega)} \|y\|.$$

We note the convergence in the Birkhoff ergodic theorem holds on a θ invariant set. The exponential pullback convergence follows easily by

$$\begin{aligned}
&\|\varphi(t,\theta_{-t}\omega,x(\theta_{-t}\omega)) - u_s(\omega)\|\\
&\le \|\varphi(t,\theta_{-t}\omega,x(\theta_{-t}\omega)) - \varphi(t,\theta_{-t}\omega,u_s(\theta_{-t}\omega))\|\\
&\le \exp\left(\int_0^t k(\theta_{-t+r}\omega)dr\right) 2 \sup_{y\in B(\theta_{-t}\omega)} \|y\|
\end{aligned}$$

which tends to zero exponentially fast.

□

As an example we consider the random evolution equation

$$\frac{dv}{dt} + (A - z(\theta_t\omega))v = e^{-z(\theta_t\omega)}F(e^{z(\theta_t\omega)}v), \quad v(0) = v_0 \in H. \tag{12}$$

where z is the unique stationary point of the one dimensional equation

$$dz + zdt = d\omega$$

and ω stands for a canonical one dimensional standard Brownian motion. Assuming for A, F the above conditions then this equation generates an RDS φ. Introduce

$$u(t) = e^{z(\theta_t\omega)}v(t) := T(\omega)v(t)$$

then u solves the spde in the Ito sense

$$du + Audt = F(u)dt + \frac{u}{2}dt + ud\omega. \tag{13}$$

In particular, the RDS

$$\psi(t,\omega,u_0) = T(\theta_t\omega)\varphi(t,\omega,T^{-1}(\omega)u_0)$$

is generated by (13).

THEOREM 25. *Suppose that $\lambda_1 > L$ then the dynamical system has a unique tempered stationary point which is contained in the ball with center zero and radius*

$$R(\omega) = \int_{-\infty}^0 e^{(\lambda_1-L)r+\int_r^0 z(\theta_q\omega)}e^{-z(\theta_r\omega)}\|F(0)\|dr.$$

PROOF. The mild solution to (12) is given by

$$v(t) = e^{\int_0^t z(\theta_r\omega)dr}S(t)v_0 + \int_0^t e^{\int_r^t z(\theta_q\omega)dq}S(t-r)e^{-z(\theta_r\omega)}F(e^{z(\theta_r\omega)}v(r))dr$$

We have that

$$\|v(t)\| \le e^{-\lambda_1 t+\int_0^t z(\theta_r\omega)dr}\|v_0\|+\int_0^t e^{-\lambda_1(t-r)+\int_r^t z(\theta_q\omega)dq}(L\|v(r)\|+e^{-z(\theta_r\omega)}\|F(0)\|)dr.$$

Applying the Gronwall lemma technique to this inequality gives us

$$\begin{aligned}\|v(t)\| \le & e^{-(\lambda_1-L)t+\int_0^t z(\theta_r\omega)dr}\|v_0\| \\ & + \int_0^t e^{-(\lambda_1-L)(t-r)+\int_r^t z(\theta_q\omega)dq}e^{-z(\theta_r\omega)}\|F(0)\|dr.\end{aligned}$$

Expressing (10) by the variation of constants formula we can conclude that the ball $B_H(0,R(\omega))$ is pullback attracting for random sets $D \in \mathcal{D}$. This ball is forward invariant. Indeed we set

$$a(\omega) = \lambda_1 - L - z(\omega), \quad f(\omega) = e^{-z(\omega)}\|F(0)\|.$$

We can show

$$\mathbb{E}z = 0, \quad \mathbb{E}\sup_{t\in[0,1]}|z(\theta_t\omega)| < \infty$$

such that $t \mapsto z(\theta_t\omega)$ grows sublinearly, see Arnold [**Arn98**] Proposition 4.1.3. Hence $\exp(-z(\omega))$ is tempered. In particular for any tempered stationary point u_s we have we have that $(\{u_s(\omega)\}_{\omega\in\Omega}) \in \mathcal{D}$. The pullback attraction then shows that $u_s(\omega) \in B_H(0,R(\omega))$. To obtain the contraction condition of Theorem 24 we obtain

$$\begin{aligned}\|v_1(t) - v_2(t)\| \le & e^{-\lambda_1 t+\int_0^t z(\theta_r\omega)dr}\|v_{10} - v_{20}\| \\ & + \int_0^t e^{-\lambda_1(t-r)+\int_r^t z(\theta_r\omega)dr}L\|v_1(r) - v_2(r)\|dr.\end{aligned}$$

The Gronwall lemma gives us the contraction constant

$$k(\omega) = -\lambda_1 + L + z(\omega)$$

with a negative expectation.

□

We note that

$$u_s(\omega) = T(\omega)v_s(\omega)$$

is the unique stationary point of the RDS ψ.

6. Random attractors

We now generalize the term global attractor for RDS.

Definition 26. *A set random invariant set A where $A(\omega)$ is compact is called random attractor for an RDS φ if*

$$\mathbb{P}\lim_{t\to\infty} d_X(\overline{\varphi(t,\omega,D(\omega))}^X, A(\theta_t\omega)) = 0 \tag{14}$$

for every random closed set such D. A is called a random pullback attractor (for random sets from $\mathcal{D}$) if we have that $A \in \mathcal{D}$ and $A(\omega)$ compact, invariant for φ and we have

$$\lim_{t\to\infty} d_X(\overline{\varphi(t,\theta_{-t}\omega,D(\theta_{-t}\omega))}^X, A(\omega)) = 0 \quad \text{for all } \omega\in\Omega,\ D\in\mathcal{D}. \tag{15}$$

For the definition of random attractors see Ochs [**Och01**], Schmalfuss [**Sch92**]. The method of random pullback attractors can be applied to obtain general random attractors. In particular, pullback attractors are well defined for a system of nonautonomous perturbations (Ω,θ) and a cocycle φ. In particular we do not need any measurable structure. It also makes sense to consider *fibred* state spaces $(H(\omega))_{\omega\in\Omega}$. In addition, we can consider other systems than closed tempered random sets $\mathcal{D}$. For instance local attractors could be modelled using another system $\mathcal{D}$, see Arnold and Schmalfuss [**AS01**].

We have the following main theorem about the existence of a random (pullback) attractor, See Crauel et al. [**CDF97**] or Keller and Schmalfuss [**KS93**] for attractors attracting random sets.

Theorem 27. *Let φ be a continuous RDS. In addition, suppose that there exists a compact pullback attracting set in $B \in \mathcal{D}$. Then φ has a random attractor which is unique in $\mathcal{D}$.*

The following corollary is straightforward, see Flandoli and Schmalfuss [**FS96**].

Corollary 28. *Let φ be a continuous RDS. In addition, suppose that there exists a compact pullback absorbing set in $B \in \mathcal{D}$. Then φ has a random attractor which is unique in $\mathcal{D}$.*

We prove the corollary.

Proof. For simplicity we also assume that the pullback absorbing set B is forward invariant. First we note that the family

$$(\varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))))_{t\ge0}$$

is decreasing: For $0\le s\le t$ we have

$$\varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)) = \varphi(s,\theta_{-s}\omega,\underbrace{\varphi(t-s,\theta_{-t}\omega,B(\theta_{-t}\omega))}_{\subset B(\theta_{-s}\omega)}).$$

Hence by the compactness assumption

$$\bigcap_{t\ge0}\varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)) \tag{16}$$

has a nonempty intersection we denote by $A(\omega)$ which is a compact set. We also have

$$\lim_{t\to\infty} d_X(\varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)),A(\omega))=0.$$

In the other case we find a sequence $(t_n)_{n\in\mathbb{N}}$ which converges to ∞, a sequence $(y_n)_{n\in\mathbb{N}}$ where $y_n\in\varphi(t_n,\theta_{-t_n}\omega,B(\theta_{-t_n}\omega))$ such that

$$d_X(y_n,A(\omega))>\epsilon$$

for some positive ϵ. By the positive invariance of B this sequence is contained in the compact set $B(\omega)$, hence there existence a converging subsequence $(y_{n'})$ with limit y. But y is contained in any of the sets we consider the intersection in (16). Hence $y\in A(\omega)$ which is a contradiction. We also have $A(\omega)\subset B(\omega)$ hence $A\in\mathcal{D}$.
The last properties show us that for any $\epsilon>0$ and $\omega\in\Omega$ there exists a $\bar{T}(\omega,\epsilon)$ such that for all $t\ge\bar{T}(\omega,\epsilon)$

$$d_X(\varphi(t,\theta_{-t}\omega,B(\theta_t\omega)),A(\omega))<\epsilon.$$

Then by the absorbing property of B we have for $s>T(\theta_{-t}\omega,D)$ that

$$\begin{aligned}\varphi(t+s,\theta_{-t-s}\omega,D(\theta_{-t-s}\omega)) =&\varphi(t,\theta_{-t}\omega,\varphi(s,\theta_{-s-t}\omega,D(\theta_{-s-t}\omega)))\\ \subset&\varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\end{aligned}$$

and hence

$$d_X(\varphi(t+s,\theta_{-t-s}\omega,D(\theta_{-t-s}\omega)),A(\omega))<\epsilon.$$

We prove the invariance of A. Straightforwardly

$$\begin{aligned}\varphi(t,\omega,A(\omega)) =&\varphi(t,\omega,\bigcap_{\tau>0}\varphi(\tau,\theta_{-\tau}\omega,B(\theta_{-\tau}\omega)))\subset\bigcap_{\tau>0}\varphi(t,\omega,\varphi(\tau,\theta_{-\tau}\omega,B(\theta_{-\tau}\omega)))\\ =&\bigcap_{\tau>0}\varphi(t+\tau,\theta_{-t-\tau}\theta_t\omega,B(\theta_{-t-\tau}\theta_t\omega))=A(\theta_t\omega).\end{aligned}$$

This follows by (16) because the family of sets forming this intersection is decreasing.
Let $x\in A(\theta_t\omega)$. Then for any $\sigma\ge0$ there exists an x_σ such that

$$x_\sigma\in\varphi(\sigma,\theta_{-\sigma}\omega,B(\theta_{-\sigma}\omega))\subset B(\omega),\quad x=\varphi(t,\omega,x_\sigma).$$

The sequence $(x_\sigma)_{\sigma\ge0}$ is relatively compact with some limit point x_0 in $A(\omega)$. By the continuity of φ we have

$$\lim_{\sigma\to\infty}\varphi(t,\omega,x_\sigma)=\varphi(t,\omega,x_0)=x$$

such that

$$\varphi(t,\omega,A(\omega))\supset A(\theta_t\omega).$$

For the uniqueness we note that for two random pullback attractors we have because of $A_1,\ A_2\in\mathcal{D}$:

$$d_X(A_1(\omega),A_2(\omega))=\lim_{t\to\infty}d_X(\varphi(t,\theta_{-t}\omega,A_1(\theta_{-t}\omega),A_2(\omega)))=0,$$

hence $A_1(\omega)\subset A_2(\omega)$. Now we exchange the rôle of A_1 and A_2.
We conclude with the measurability of A. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of random variables such that

$$\overline{\bigcup_{n\in\mathbb{N}}\{x_n(\omega)\}}^X=B(\omega)\quad\text{for }\omega\in\Omega.$$

Then

$$\varphi(t,\omega,\overline{\bigcup_{n\in\mathbb{N}}\{x_n(\omega)\}}^X) \subset \overline{\bigcup_{n\in\mathbb{N}}\varphi(t,\omega,\{x_n(\omega)\})}^X \tag{17}$$

by the continuity of $\varphi(t,\omega,\cdot)$. To see the inverse inclusion we note that for a y in the right hand side of (17) we have a converging sequence $x_{n'}(\omega)$ such that this sequence has a limit $x_0(\omega)$ which follows by the compactness of $B(\omega)$ and

$$\lim_{n\to\infty}\varphi(t,\omega,x_{n'}(\omega)) = y, \quad y = \varphi(t,\omega,x_0(\omega)),$$

hence

$$\varphi(t,\omega,\overline{\bigcup_{n\in\mathbb{N}}\{x_n(\omega)\}}^X) \supset \overline{\bigcup_{n\in\mathbb{N}}\varphi(t,\omega,\{x_n(\omega)\})}^X.$$

Since $\varphi(t,\omega,x_n(\omega))$ is a random variable, the right hand side and hence the left hand side is a random set. Since $\varphi(n,\theta_{-n}\omega,B(\theta_{-n}\omega))$ is decreasing

$$d_X(z,A(\omega)) = \lim_{n\to\infty} d_X(z,\varphi(n,\theta_{-n}\omega,B(\theta_{-n}\omega)))$$

where the right hand side is measurable. □

We study the following application. We consider the evolution equation (9). We assume again that A is a linear unbounded operator satisfying the usual conditions. F is assumed to be Lipschitz continuous which ensures the existence of an RDS φ on H. ω is again a trace class Brownian motion in H. Let us also assume that

$$\|F(u)\| \le l\|u\| + c_0$$

In general the constant l can be chosen smaller that L. We assume $l \le \lambda_1$. Considering the weak solution to (9) then we have the estimate

$$\|v(t)\| \le e^{-\lambda_1 t}\|v_0\| + \int_0^t e^{-\lambda_1(t-r)}(l\|v(r)\| + l\|Z(\theta_r\omega)\| + c_0)dr$$

Comparing this norm with the solution of (11) we have for the RDS generated by v

$$\limsup_{t\to\infty}\sup_{x\in D(\theta_{-t}\omega)}\|\varphi(t,\theta_{-t}\omega,x)\| \le w_s(\omega) + 1$$

where w_s is the stationary point to (10) with

$$a = \lambda_1 - l, \quad f(\omega) = l\|Z(\omega)\| + c_0.$$

Hence The ball $C(\omega)$ with center zero and radius $2w_s(\omega)$ is a pullback absorbing set and forward invariant set. In addition by the temperedness of v_s this ball is in $\mathcal{D}$. We claim that

$$\overline{\varphi(1,\theta_{-1}\omega,C(\theta_{-1}\omega)))} =: B(\omega) \subset C(\omega)$$

is a random compact pullback absorbing set. Since C is pullback absorbing so is B. B is in $\mathcal{D}$ and in particular a random set. Indeed we have for some sequence of random variables $(y_n)_{n\in\mathbb{N}}$ such that

$$\overline{\bigcup_{n\in\mathbb{N}}\{y_n(\omega)\}}^X = C(\omega)$$

the relation

$$\overline{\varphi(1,\omega,\bigcup_{n\in\mathbb{N}}\{y_n(\omega)\})}^X = \overline{\bigcup_{n\in\mathbb{N}}\{\varphi(1,\omega,y_n(\omega))\}}^X \subset \overline{\varphi(1,\omega,C(\omega))}^X$$
$$\subset \overline{\varphi(1,\omega,\overline{\bigcup_{n\in\mathbb{N}}\{y_n(\omega)\}}^X)}^X = \overline{\varphi(1,\omega,\bigcup_{n\in\mathbb{N}}\{y_n(\omega)\})}^X.$$

To see that $\varphi(1,\omega,\cdot)$ is a compact mapping we note that $D(A^\alpha)\subset H$ is a compact embedding for $\alpha>0$. By Lemma 12 we obtain

$$\begin{aligned}\sup_{v_0\in B(\omega)}\|A^\alpha\varphi(1,\omega,v_0)\| \le{}& c_\alpha 2w_s(\omega)\\ &+\int_0^1\|A^\alpha S(1-r)\|(l\sup_{v_0\in C(\omega)}\|\varphi(r,\omega,v_0)\|+l\|Z(\theta_r\omega)\|+c)dr\\ \le{}&2c_\alpha w_s(\omega)\\ &+\int_0^1\frac{c_\alpha}{(1-r)^\alpha}(l\sup_{v_0\in C(\omega)}\|\varphi(r,\omega,v_0)\|+l\|Z(\theta_r\omega)\|+c)dr\end{aligned}$$

By the Gronwall lemma

$$\sup_{r\in[0,1]}\sup_{v_0\in C(\omega)}\|\varphi(r,\omega,v_0)\|\le c(\omega)<\infty\quad\text{for } D\in\mathcal{D}.$$

For $\alpha<1$ the integral exists.

We are interested to find a random attractor for the RDS ψ generated by (3). We just know that this system is conjugated to φ. Indeed we have the random attractor for ψ denoted by A'. This attractor is defined by

$$A'(\omega)=T(\omega,A(\omega)).$$

In particular, we can show that by the temperedness of $\|Z(\omega)\|$:

$$(T(\omega,D(\omega)))_{\omega\in\Omega},\quad (T^{-1}(\omega,D(\omega)))_{\omega\in\Omega}\in\mathcal{D}.$$

7. Random invariant manifolds

Given an RDS φ over a Hilbert space H. We are interested in finding forward invariant random sets in H which have the structure of a manifold with some regularity. For instance this manifold can be given by a (random) graph of a Lipschitz continuous mapping. To construct such a manifold we have to split the state space H into particular subspaces:

$$H=H^+\oplus H^-.$$

The projections onto these spaces are denoted by P^+ and P^-. In the next section we will give some ideas how do deal with the case of an ω depending splitting. The manifolds we consider should describe the stability behavior of the RDS φ.

Definition 29. *The ω-dependent sets $M=(M(\omega))_{\omega\in\Omega}$ are called random Lipschitz manifolds if*

$$M(\omega)=\{(x^+,m(\omega,x^+)):x^+\in H^+\}$$

where

$$m:\Omega\times H^+\mapsto H^-$$

which is a Carathéodory mapping. In addition the mapping

$$x^+ \mapsto m(\omega, x^+)$$

is assumed to be Lipschitz continuous.

Such a random manifold is called global because the domain of the graph of the function m is given over H^+. This manifold is a random set. In some situations it is also interesting to consider random manifold given by a graph with a domain which is a subset of H^+.

Lemma 30. *A random manifold is a closed random set.*

Proof. We have

$$\operatorname{dist}(x, M(\omega)) = \inf_{y \in E^+} \|x - (y + m(\omega, y))\| = \inf_{y \in Q^+} \|x - (y + m(\omega, y))\|$$

where Q^+ is a countable dense set in E^+ by the continuity of $m(\omega, \cdot)$. But the infimum of countable many random variables is measurable. □

A version of these random invariant manifolds are so called inertial manifolds. These manifolds are forward invariant with respect to the dynamics of φ and attract all states of the system:

$$\begin{aligned} &\varphi(t, \omega, M(\omega)) \subset M(\theta_t\omega) \quad \text{for all } t \geq 0, \omega \in \Omega, \\ &\lim_{t\to\infty} d_H(\varphi(t, \omega, x), M(\theta_t\omega)) = 0 \\ &\quad \text{with exponential speed, for all } t \geq 0, \omega \in \Omega, \quad x \in H. \end{aligned}$$

Similar to random attractors, random inertial manifolds describe the long term behavior of φ. In particular, the essential behavior of the system takes place in a neighborhood of φ. These manifolds also play some rôle to reduce asymptotically the dimension of the system. Suppose the RDS with inertial manifold M is generated by the random evolution equation

$$\frac{dv}{dt} = Av + F(\theta_t\omega, v) \tag{18}$$

where $F(\omega, \cdot)$ has the Lipschitz constant $L(\omega)$ and $t \mapsto L(\theta_t\omega)$ is supposed to be integrable on any bounded interval. Without proof we mention that these conditions ensure the existence of a unique global solution which depends measurably on ω and continuously on the initial condition. Hence this solution generates an RDS. Suppose that the spaces H^+ and H^- are given by the span of eigenelements of A related to eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ and $\lambda_{N+1} \cdots$ where $\lambda_N > \lambda_{N+1}$. Now we consider the finite dimensional random differential equation

$$\frac{dv^+}{dt} = Av^+ + F(\theta_t\omega, v^+ + m(\theta_t\omega, v^+))$$

which has a unique global solution. This solution generates an RDS on the finite dimensional space H^+. Indeed for existence and uniqueness we use that m is Lipschitz continuous in the second argument. Now on the manifold to obtain the behavior of the H^- part we do not have to solve a differential equation, we simply consider $v^-(t) = m(\theta_t\omega, v^+(t))$.

Let us consider (18). We consider the operator A with a spectrum $\lambda_1 \geq \lambda_2 \geq \cdots$ tending to $-\infty$. The eigenvectors to these eigenvalues form a complete orthonormal system for H. Assume that F has a Lipschitz constant $L(\omega)$ uniformly bounded by

$\bar{L}$. In addition, F is uniformly bounded. The main assumption for the following is the *gap condition*. Let $\hat{\lambda} > \check{\lambda}$ two neighbored eigenvalues of A such that

$$\hat{\lambda} - \check{\lambda} - 4\bar{L} > 0.$$

Let us write for the following $S^\pm$, $F^\pm$ for $P^\pm S$ and $P^\pm F$. Then $P^\pm$ are orthoprojections having the operator norm one. Considering S^- we have the estimate

$$\|S^-(t)\|_{L(H)} \le e^{\hat{\lambda} t} \quad \text{for } t \ge 0.$$

Similar we have

$$\|S^+(t)\|_{L(H)} \le e^{\check{\lambda} t} \quad \text{for } t \le 0.$$

We now formulate a theorem for the existence of random invariant manifolds, see also Lu et al. [DLS04], [DLS03] or Schmalfuss [Sch05]:

THEOREM 31. *Suppose that the gap condition holds for the system given by* (18). *Then the RDS has a random invariant manifold.*

PROOF. (1) We consider the Banach space

$$(\mathcal{H}, \|\!|\cdot|\!\|), \quad \mathcal{H} = \{y : \mathbb{R}^- \to H, \text{ continuous } \|\!|y|\!\| = \sup_{t\in\mathbb{R}^-} e^{\mu t}\|y(t)\|\}.$$

Later we will set $\mu = -(\hat{\lambda} + \bar{\lambda})/2$.

(2) We introduce the Lyapunov Perron transform. For fixed $\omega \in \Omega$, $v_0^+ \in H^+$

$$\begin{aligned}\mathcal{T}_{v_0^+,\omega}(y)[t] =& S^+(t)v_0^+ - \int_t^0 S^+(t-r)F^+(\theta_r\omega, y(r))dr \\ &+ \int_{-\infty}^t S^-(t-r)F^-(\theta_r\omega, y(r))dr, \quad t \le 0.\end{aligned}$$

This mapping has the following two properties.

$$\begin{aligned}&\mathcal{T}_{v_0^+,\omega} : \mathcal{H} \to \mathcal{H}, \\ &\left|\!\left|\!\left|\mathcal{T}_{v_0^+,\omega}(y_1) - \mathcal{T}_{v_0^+,\omega}(y_2)\right|\!\right|\!\right| \le k \,\|\!|y_1 - y_2|\!\| \quad \text{for all } y_1,\, y_2 \in \mathcal{H}\end{aligned}$$

for a constant $k < 1$. We now prove only the second property. We can estimate

$$\begin{aligned}&\sup_{t\le 0} e^{\mu t}\Big\| - \int_t^0 S^+(t-r)(F^+(\theta_r\omega, y_1(r)) - F^+(\theta_r\omega, y_1(r)))dr \\ &\qquad + \int_{-\infty}^t S^-(t-r)(F^-(\theta_r\omega, y_1(r)) - F^-(\theta_r\omega, y_1(r)))dr\Big\| \\ &\le \sup_{t\le 0}\int_t^0 e^{\mu(t-r)}e^{\hat{\lambda}(t-r)}e^{\mu r}\bar{L}\|y_1(r) - y_2(r)\|dr \\ &\qquad + \int_{-\infty}^t e^{\mu(t-r)}e^{\check{\lambda}(t-r)}e^{\mu r}L\|y_1(r) - y_2(r)\|dr \\ &\qquad \le \bar{L}\,\|\!|y_1 - y_2|\!\|\left(\sup_{t\le 0}\int_t^0 e^{(\hat{\lambda}+\mu)(t-r)}dr + \sup_{t\le 0}\int_{-\infty}^t e^{(\check{\lambda}+\mu)(t-r)}dr\right).\end{aligned}$$

Calculating the integrals the integrals in the chain of inequalities we have the estimate

$$\bar{L}\left(\frac{2}{\hat{\lambda} - \check{\lambda}} + \frac{2}{\hat{\lambda} - \check{\lambda}}\right)\|\!|y_1 - y_2|\!\|$$

where the two first factors for the contraction constant k which is smaller than one by the gap condition. We denote the unique fixed point in $\mathcal{H}$ by $\Gamma(\omega, v_0^+)$.

We prove that the mapping

$$H^+ \ni v_0^+ \mapsto \Gamma(\omega, v_0^+) \in \mathcal{H}$$

is Lipschitz continuous. We have

$$\begin{aligned}
&\left\|\left|\Gamma(\omega, v_{01}^+) - \Gamma(\omega, v_{02}^+)\right\|\right| = \left\|\left|\mathcal{T}_{v_{01}^+,\omega}(\Gamma(\omega, v_{01}^+)) - \mathcal{T}_{v_{02}^+,\omega}(\Gamma(\omega, v_{02}^+))\right\|\right| \\
&\le \left\|\left|\mathcal{T}_{v_{01}^+,\omega}(\Gamma(\omega, v_{01}^+)) - \mathcal{T}_{v_{02}^+,\omega}(\Gamma(\omega, v_{01}^+))\right\|\right| + \left\|\left|\mathcal{T}_{v_{02}^+,\omega}(\Gamma(\omega, v_{01}^+)) - \mathcal{T}_{v_{02}^+,\omega}(\omega, \Gamma(v_{02}^+))\right\|\right| \\
&\le \sup_{t\le 0} e^{\mu t}\|S^+(t)\|_{L(H)}\|v_{01}^+ - v_{02}^+\| + k \left\|\left|\Gamma(\omega, v_{01}^+) - \Gamma(\omega, v_{01}^+)\right\|\right|
\end{aligned}$$

We have

$$e^{\mu t}\|S^+(t)\|_{L(H)} \le e^{(\mu+\hat\lambda)t} = e^{\frac{\hat\lambda-\check\lambda}{2}t} \to 0 \quad \text{for } t \to -\infty.$$

such that $S^+(\cdot)v_0^+ \in \mathcal{H}$. From the above chain of inequalities we obtain

$$\left\|\left|\Gamma(v_{01}^+, \omega) - \Gamma(v_{02}^+, \omega)\right\|\right| \le L_M \left\|\left| y_1 - y_2\right\|\right|, \quad L_M = \frac{\sup_{t\le 0} e^{\mu t}\|S^+(t)\|_{L(H)}}{1-k}.$$

Now we can define a Lipschitz manifold $M = (M(\omega))_{\omega\in\Omega}$ given by the graph

$$m(\omega, x^+) = P^-\Gamma(x^+, \omega), \quad x^+ \in H^+.$$

To see this we have to show that

$$\varphi(T, \omega, M(\omega)) \subset M(\theta_T\omega).$$

But this follows if we can show that

$$\Xi_T(\sigma, \omega) = \begin{cases} \Gamma(v_0^+, \omega)[\sigma + T] & : \quad \sigma < T \\ \varphi(\sigma + T, \omega, \Gamma(v_0^+, \omega)[0]) & : \quad \sigma \in [-T, 0] \end{cases}$$

is fixed point of the Lyapunov Perron Transform $\Gamma(P^+\varphi(T, \omega, v_0), \theta_T\omega)[\sigma]$ when we replace v_0^+ by $P^+\varphi(T, \omega, v_0)$ and ω by $\theta_T\omega$ then

$$P^-\Xi_T(0, \omega) = m(\theta_T\omega, P^+\varphi(T, \omega, \Gamma(v_0^+, \omega)[0]))$$

and hence

$$\varphi(T, \omega, \Gamma(v_0^+, \omega)[0])) \in M(\theta_T\omega)$$

giving the equivalence. For the proof we only consider the E^- component. We have

$$\begin{aligned}
P^-\varphi(t, \omega, \Gamma(v_0^+, \omega)[0]) =& S^-(t)\int_{-\infty}^0 S^-(-r)F(\theta_r\omega, \Gamma(v_0^+, \omega)[r])dr \\
&+ \int_0^t S(t-r)F(\theta_r\omega, \varphi(r, \omega, \Gamma(v_0^+, \omega)[r])dr.
\end{aligned}$$

Replace t by $\sigma+T$ for $\sigma\geq -T$. Then we have

$$\begin{aligned}P^-\Xi_T(\sigma,\omega)&=P^-\varphi(\sigma+T,\omega,\Gamma(v_0^+,\omega)[0])\\&=\int_{-\infty}^0 S^-(\sigma+T-r)F(\theta_r\omega,\Gamma(v_0^+,\omega)[r])dr\\&\quad+\int_0^{\sigma+T} S^-(\sigma+T-r)F(\theta_r\omega,\varphi(r,\omega,\Gamma(v_0^+,\omega)[r])dr\\&=\int_{-\infty}^{-T} S^-(\sigma-r)F(\theta_{r+T}\omega,\Gamma(v_0^+,\omega)[r+T])dr\\&\quad+\int_{-T}^{\sigma} S^-(\sigma-r)F(\theta_{r+T}\omega,\varphi(r,\omega,\Gamma(v_0^+,\omega)[r])dr\\&=\int_{-\infty}^{\sigma} S^-(\sigma-r)F(\theta_r\theta_t\omega,\Xi_T(r,\omega))dr\end{aligned}$$

which is the E^- component of the fixed point equation for $\sigma\in[-T,0]$. Let us consider $\sigma<-T$:

$$\begin{aligned}&\int_{-\infty}^{\sigma} S^-(\sigma-r)F(\theta_{r+T}\omega,\Xi_T(r,\omega))dr\\&=\int_{-\infty}^{\sigma} S^-(\sigma+T-r)F(\theta_r\omega,\Gamma(v_0^+,\omega)[r])dr=P^-\Gamma(v_0^+,\omega)[\sigma+T]=P^-\Xi_T(\sigma,\omega)\end{aligned}$$

such that Ξ_T satisfies the E^- component of the Lyapunov Perron transform. Similar we can show the for the E^+ component. □

We can consider unstable manifolds. Suppose that (18) has the stationary point $u_s=0$. Then the set of all states converging to zero for $t\to\infty$ is called the stable set. Sometimes we consider the set of states converging to zero with a particular exponential speed. Often then these sets can be expressed by a random invariant manifold. Similar considering $t\to-\infty$ we could consider the unstable set. In case that our RDS is generated by a random or stochastic parabolic differential equation in general it is not possible to obtain solutions for $t\leq 0$. Now the term of entire trajectory comes into the play. We call a family of random variables $(x_t)_{t\in\mathbb{R}}$ such that $x_t(\omega)\in H$ a complete trajectory for the cocycle φ if

$$\varphi(s,\theta_t\omega,x_t(\omega))=x_{t+s}(\omega)\quad\text{for all } t\in\mathbb{R},\ s\in\mathbb{R}^+,\ \omega\in\Omega.$$

Now $M(\omega)$ is called the unstable set if for every $x\in\ M(\omega)$ we have a complete trajectory $(x_t)_{\in\mathbb{R}}$ with $x_0(\omega)=x$ and

$$\lim_{t\to-\infty}x_t(\omega)=0.$$

We can now apply the Lyapunov Perron method to obtain a random unstable manifold. We assume that the eigenfunctions of A form a complete orthonormal system of A but the first N eigenvalues are positive. In particular, we assume that the space H^+ is spanned by the first N' eigenfunctions of A where $N'\leq N$. Assuming that the gap condition is satisfied

$$\lambda_{N'}-\lambda_{N'+1}-4\bar{L}>0.$$

We assume that

$$\mu=-(\lambda_{N'+1}+\lambda_{N'})/2=-(\check{\lambda}+\hat{\lambda})/2<0.$$

This inequality holds for $N' > N$. However when $N' = N$ this inequality has not to be satisfied. In particular if $\lambda_{N+1} < 0$, μ could be positive. To avoid this problem in the definition of μ we replace the value λ_{N+1} by some $\check{\lambda}$ such that $0 > \check{\lambda} \geq \lambda_{N+1}$. However then we reduce the set of nonlinearities F satisfying the gap condition. Then we can construct a random invariant manifold by the fixed point $\Gamma(\omega, v_0^+)$ of the Lyapunov Perron operator. In particular, we have a complete trajectory

$$x(t,\omega) = \begin{cases} \Gamma(\omega, v_0^+)[t] & : \quad t \leq 0, \\ \varphi(t,\omega,\Gamma(\omega, v_0^+)[0]) & : \quad t \geq 0. \end{cases}$$

Since we are looking for mild solutions we have to check that

$$\begin{aligned} S^+(s)\Bigg(S^+(t)v_0^+(\omega) - \int_t^0 S^+(t-r)F^+(\theta_r\omega, \Gamma(v_0^+(\omega),\omega)[r])dr \\ + \int_{-\infty}^t S^-(t+s-r)F^-(\theta_r\omega, \Gamma(v_0^+(\omega),\omega)[r])dr\Bigg) \\ + \int_0^s S(s-r)F(\theta_{r+t}\omega, \Gamma(v_0^+(\omega),\omega)[r+t])dr. \end{aligned}$$

Let us consider the E^+ component. We obtain by a linear integral substitution

$$\begin{aligned} S^+(s+t)v_0^+(\omega) - \int_t^0 S^+(t+s-r)F^+(\theta_r\omega, \Gamma(v_0^+(\omega),\omega)[r])dr \\ + \int_t^{s+t} S^+(\theta_r\omega, \Gamma(v_0^+(\omega),\omega)[r])dr = P^+\Gamma(v_0^+(\omega),\omega)[s+t] \\ = P^+x_{t+s}(\omega) \end{aligned}$$

and similar for the E^- component.

Since $\mu < 0$ he Banach space $\mathcal{H}^-$ which depends on the parameter μ contains only paths with an exponential decay to zero related to μ which then proves that the manifold M is unstable.

8. The Oseledets Theorem

In this section we consider the stability behavior of linear random dynamical systems. The main result about this topic is the Oseledets theorem (multiplicative ergodic theorem). We present here some details. A complete presentation can be found in Arnold [?Arn96] Chapter 3.

DEFINITION 32. *A measurable mapping*

$$\Phi : \mathbb{T} \times \Omega \mapsto \mathbb{R}^{d\times d}$$

is called a linear RDS if

$$\begin{aligned} \Phi(t+s,\omega) =& \Phi(t,\theta_s\omega)\Phi(s,\omega) \\ \Phi(0,\omega) =& \mathrm{Id}. \end{aligned}$$

The composition here is the composition of two linear operators. Here $\mathbb{T}$ can be $\mathbb{Z}^+$, $\mathbb{R}^+$ or $\mathbb{Z}$, $\mathbb{R}$. For the later cases we have that

$$\Phi(t,\omega)^{-1} = \Phi(-t,\theta_t\omega).$$

At first we introduce Lyapunov exponents and Lyapunov indices for time dependent function which are useful to define random invariant (stable or unstable) manifolds.

DEFINITION 33. *Let f be a mapping from $\mathbb{Z}^+$ or $\mathbb{R}^+$ to $\mathbb{R}^d$. Then*

$$\hat{\lambda}(f) = \limsup_{t\to\infty} \frac{1}{t} \log \|f(t)\| \in \bar{\mathbb{R}}$$

is called the Lyapunov index.

A very simple example is given by

$$f(t) = ce^{\lambda t}, \quad c \neq 0$$

such that $\hat{\lambda}(f) = \lambda$. This parameter characterized the averaged exponential growth of a function f. Here we have some properties of the Lyapunov index.

REMARK 34. *We have*

- $\hat{\lambda}(c) = 0$, *for* $c \neq 0$.
- *For* $c = 0$ *we set* $\hat{\lambda}(0) := -\infty$.
- $\hat{\lambda}(cf) = \hat{\lambda}(f)$ *for* $c \neq 0$.
- $\hat{\lambda}(f+g) \leq \max(\hat{\lambda}(f), \hat{\lambda}(g))$ *with equality when* $\lambda(f) \neq \lambda(g)$ *where "* $<$ *" is possible for instance for* $f = -g$.

Let $\Phi : \mathbb{T} \times \mathbb{R}^d$ be a linear dynamical system or more general the solution mapping of a linear nonautonomous differential equation. We define $\mathbb{T}$ to be $\mathbb{Z}^+$ or $\mathbb{R}^+$ or $\mathbb{Z}$ or $\mathbb{R}$.

DEFINITION 35. *For $x \in \mathbb{R}^d$ the numbers*

$$\lambda(x) = \hat{\lambda}(\Phi(t)x) = \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi(t)x\| = \lambda^+(x),$$

$$\lambda^-(x) = \hat{\lambda}(\Phi(-t)x) = \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi(-t)x\|$$

are called Lyapunov exponents.

The numbers λ^+, λ^- inherit the properties of $\hat{\lambda}$. Indeed, we have

- $\lambda^\pm(x) \in \mathbb{R} \cup \{-\infty, \infty\}$ and $\lambda^\pm(0) = -\infty$,
- $\lambda^\pm(\alpha x) = \lambda^\pm(x)$ for $\alpha \neq 0$,
- $\lambda^\pm(x+y) \leq \max(\lambda^\pm(x), \lambda^\pm(y))$ with equality when $\lambda^\pm(x) \neq \lambda^\pm(y)$.

We have the following interesting Lemma:

LEMMA 36. *The Lyapunov exponent can only have at most d different values: Let $p \leq d$ be the number of (different) Lyapunov exponents*

$$-\infty \leq \lambda_p < \lambda_{p-1} < \cdots < \lambda_1 < \infty.$$

PROOF. We consider the sets

$$V_\lambda = \{x \in \mathbb{R}^d : \lambda(x) \leq \lambda\}.$$

By Remark 34 these sets are linear subspaces of $\mathbb{R}^d$. In particular we can have only finitely many of these spaces

$$\mathbb{R}^d = V_1 := V_{\lambda_1} \supset V_2 := V_{\lambda_2} \supset \cdots \supset V_p := V_{\lambda_p} \supset V_{p+1} = \{0\} \tag{19}$$

with proper inclusion such that

$$d = \mathrm{dim} V_1 > \mathrm{dim} V_2 > \cdots > \mathrm{dim} V_{p+1} = 0.$$

□

From this property we can derive that we have $\lambda(x) = \lambda_i$ if and only if $x \in V_i \setminus V_{i+1}$. Indeed, if $\lambda(x) < \lambda_i$ then by the finite number of Lyapunov exponents we have $\lambda(x) \le \lambda_{i+1}$ such that $x \in V_{i+1}$. A sequence of spaces given in (18) is called a filtration or flag of subspaces of $\mathbb{R}^d$. The numbers $d_i = \mathrm{dim} V_i - \mathrm{dim} V_{i+1}$ are called the multiplicity of λ_i.

Let now Φ be defined for $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$. Similar we can consider the case $t \to -\infty$. In particular we have $p^- \le d$ Lyapunov exponents related to linear spaces $V_i^- \subset \mathbb{R}^d$

$$\begin{aligned}
&\lambda^-(x) = \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi(-t)x\| = \limsup_{t\to-\infty} \frac{1}{-t} \log \|\Phi(t)x\| \\
&-\infty \le \lambda^-_{p^-} < \lambda^-_{p^- -1} < \cdots < \lambda^-_1 < \infty, \\
&\mathbb{R}^d = V_1^- := V_{\lambda_1^-}^- \supset V_2^- := V_{\lambda_2^-}^- \supset \cdots \supset V_p^- := V_{\lambda_{p^-}} \supset V^-_{p^-+1} = \{0\}, \\
&\lambda^-(x) = \lambda_i^- \quad \text{if and only if } x \in V_i^- \setminus V_{i+1}^-
\end{aligned}$$

where $d_i^- = \mathrm{dim} V_i^- - \mathrm{dim} V_{i+1}^-$ are called the multiplicity of λ_i^-.

Let us assume

(20) $$p = p^-, \quad d_i = d^-_{p-i+1}, \lambda_i = -\lambda^-_{p-i+1}, \quad i = 1, \cdots, p$$

and in addition

(21) $$V_{i+1} \cap V^-_{p-i+1} = \{0\}$$

such that

$$\mathrm{dim} V_{i+1} + \mathrm{dim} V^-_{p-i+1} = d.$$

We now can split $\mathbb{R}^d$

$$E_i = V_i \cap V^-_{p-i+1}, \quad \mathbb{R}^d = E_1 \oplus \cdots \oplus E_p.$$

In particular, we can derive a dynamical splitting of $\mathbb{R}^d$:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t)x\| = \lambda_i \quad \text{if and only if } x \in E_i \setminus \{0\}.$$

Indeed, we have $\lambda^+(x) \le \lambda_i$ and $-\lambda(x) = \lambda^-(x) \le \lambda^-_{p+1-i} = -\lambda_i$ and hence $\lambda_i = \lambda(x)$.

LEMMA 37. *Let Φ a onesided (nonrandom) dynamical system (such that we have not to define the spaces V_i^-). Then we have the inclusion*

$$\Phi(s) V_{\lambda_i} \subset V_{\lambda_i}.$$

for $t \ge 0$ and analogue for $t \le 0$.

Indeed let $x \in V_i$ for $s \ge 0$

$$\begin{aligned}
\hat\lambda(\Phi(s)x) =& \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi(t)\Phi(s)x\| \\
& \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi(t+s)x\| = \limsup_{t\to\infty} \frac{1}{t+s} \log \|\Phi(t+s)x\| = \hat\lambda(x)
\end{aligned}$$

such that $\Phi(s)x \in V_{\lambda_i}$.

Let us remark if the linear operators $\Phi(t)$ have full rank then the inclusion in the statement can be replaced by equality.

As a preparation to obtain an idea about the proof of the Oseledets theorem we now introduce a tool form linear algebra. We consider singular value decomposition of a $d \times d$ matrix.

Let $A \in \mathbb{R}^{d\times d}$ then there exist $U, V \in O(d)$[2] and numbers $\delta_1 \geq \cdots \geq \delta_d \geq 0$ such that

$$A = VDU, \quad D = \text{diag}(\delta_1, \cdots, \delta_d).$$

The numbers δ_i are called the singular values of A. We need the following properties

- The singular values of A are the eigenvectors of $(A^TA)^{\frac{1}{2}}$, $(AA^T)^{\frac{1}{2}}$.
- The columns of U^T are the associated eigenvectors.
- $\delta_1(A) = \delta_1 = \|A\|$,
- $|\det A| = \delta_1(A) \cdot \ldots \cdot \delta_d(A)$.
- If A has an inverse then $\delta_i(A^{-1}) = 1/\delta_{d+1-i}(A)$ for $i = 1, \cdots, d$.

Definition 38. *A (measurable) random process X is called subadditive if*

$$X(t+s,\omega) \leq X(s,\omega) + X(t,\theta_s\omega), \quad t, s \geq 0.$$

An example for a process which is (sub)additive is the following integral

$$X(t,\omega) = \int_0^t g(\theta_r\omega)dr$$

when g is sufficiently regular such that the above integral defines a random variable. As a generalization of the Birkhoff ergodic theorem we have

Theorem 39. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a metric dynamical system with time set $\mathbb{N}$. (We do not assume in general that $\mathbb{P}$ is ergodic.) We consider the subadditive process X with $X(1)^+ \in L_1(\Omega)$. Then there exists a $\theta = (\theta_n)_{n\in\mathbb{N}}$ forward invariant set of full measure $\tilde{\Omega} \subset \mathcal{F}$*

- $\lim_{n\to\infty} \frac{1}{n} X(n,\omega) = \tilde{X}(\omega)$, $\tilde{X}(\omega) \in \mathbb{R} \cup \{-\infty\}$ *and* $\tilde{X}(\omega) = \tilde{X}(\theta_n\omega)$.
- $\tilde{X}(\omega) = \gamma = \inf_{n\in\mathbb{N}} \frac{1}{n} X(n,\omega)$ *if $\mathbb{P}$ is ergodic.*
- $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}X(n,\omega) = \gamma, \quad$ *if* $X(1) \in L_1(\omega)$.

Suppose the linear dynamical system Φ is given by $u(t) = \Phi(t)u_0 \in \mathbb{R}^d$ where u solves the initial value problem

$$u' = Au$$

with initial condition $u(0) = x$ and $A \in \mathbb{R}^{d\times d}$. Consider this problem over the complex field $\mathbb{C}$. Then it is known you can split $\mathbb{C}^d$ into $p'(\leq d)$ linear subspaces F_i of $\mathbb{C}^d$ given by the algebraic eigenspaces related to the eigenvalues $\mu_1, \cdots, \mu_{p'}$. Since the matric A is real and the eigenvalues of A appear in conjugated complex form it makes sense to study only real solutions. Then it makes sense to consider the splitting

$$\mathbb{R}^d = E_1 \oplus \cdots \oplus E_p \quad (p \leq p')$$

where E_i is formed by the real elements of $F_{i'}$ determined by eigenvalues with the same real part. For details we refer to Amann [**Ama83**] Chapter 3.

[2] $O(d)$ the group of orthogonal matrices in $\mathbb{R}^{d\times d}$.

Consider now a linear RDS $\Phi = \Phi(t,\omega)$ on $\mathbb{R}^d$ with time set $\mathbb{T} = \mathbb{R}$. The family $\Phi(t,\omega)$ consists of measurable linear operators satisfying the cocycle property. For instance such an RDS can by generated by the solution of

$$\frac{du}{dt} = A(\theta_t \omega)u.$$

The dynamics of the RDS Φ is given by the Oseledets theorem (multiplicative ergodic theorem). In particular, considering the (i.g. complex) eigenvalues of $A(\omega)$ does not allow in general to find a random version of the splitting of $\mathbb{R}^d$. The adequate objects which allow to solve this problems are the singular values of the linear RDS Φ.

We present the following special form of the multiplicative ergodic theorem.

THEOREM 40. *Let Φ be a linear RDS with values in $\mathbb{R}^d$ over a ergodic metric dynamical system. Suppose that the integrability condition holds*

$$\mathbb{E} \sup_{0\le t\le 1} \log^+ \|\Phi(t,\omega)^{\pm 1}\| < \infty.$$

Then there exists a θ invariant set Ω' of full measure and numbers

$$d_1, \dots, d_p, \qquad \lambda_1 > \cdots > \lambda_p, \quad p \le d.$$

In addition there exists a random splitting of $\mathbb{R}^d$

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_p(\omega)$$

and measurable projections $P_j(\omega)$ onto these spaces. The space $E_j(\omega)$ have dimension d_j independent of ω. Moreover, we have.

$$\Phi(t,\omega)E_j(\omega) = E_j(\theta_t\omega) \quad \textit{for } t \in \mathbb{R}, \quad ,$$

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)x\| = \lambda_j \quad \textit{if and only if } x \in E_j(\omega) \setminus \{0\}$$
$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\| = \lambda_1 = \mathbb{E} \log \|\Phi(1,\omega)\|$$

for $\omega \in \Omega'$.

We would like to present a few ideas about the proof of this theorem. However we consider only the case when the state space is $\mathbb{R}^2$ and the time set $\mathbb{T}$ is given by $\mathbb{Z}$. Then we can read the integrability condition of the last theorem

$$\mathbb{E} \log^+ \|\Phi(1,\omega)^{\pm 1}\| < \infty.$$

For the proof we refer to Arnold [Arn98] Chapter 3.

First we note that

$$X_1(n,\omega) = \log \|\Phi(n,\omega)\| = \delta_1(\Phi(n,\omega)), \quad n \in \mathbb{N}$$
$$X_2(n,\omega) = \log |\det\Phi(n,\omega)| = \delta_1(\Phi(n,\omega))\delta_2(\Phi(n,\omega)), \quad n \in \mathbb{N}$$

are subadditive processes. The integrability condition then allows to apply Theorem 39 (the Furstenberg Kersten theorem) to these processes. Then these limits are invariant functions γ_1, γ_2. We set $\gamma_1(\omega) = \lambda_1(\omega)$ and $\lambda_2(\omega) = \gamma_2(\omega) - \lambda_1(\omega)$. Let us assume for the following $\infty > \lambda_1(\omega) > \lambda_2(\omega) > -\infty$. We can apply the Furstenberg Kesten for $\mathbb{T} = \mathbb{N}^-$. We obtain invariant random variables $\lambda_1^-(\omega)$ and

$\lambda_2^-(\omega)$. By the relation of $\Phi(n,\omega)$ and $\Phi(n,\omega)^{-1}$ and that $\lambda_1(\omega) = -\lambda_2^-(\omega)$ and $\lambda_2(\omega) = -\lambda_1^-(\omega)$ on a θ invariant set of full measure. Indeed, we have

$$\|\Phi(-n,\omega)\| = \delta_1(\Phi(-n,\omega)) = \delta_1(\Phi(n,\theta_{-n}\omega)^{-1}) = \delta_2^{-1}(\Phi(n,\theta_{-n}\omega))$$

such that

$$\begin{aligned}\frac{1}{n}\log\|\Phi(-n,\omega)\| &= \frac{1}{n}\log\delta_2^{-1}(\Phi(n,\theta_{-n}\omega))\\ &= -\frac{1}{n}\log\delta_2(\Phi(n,\theta_{-n}\omega)).\end{aligned}$$

The left hand side and hence the right hand side defines a subadditive process such that there exists a limit for $n\to\infty$ given by $\lambda_1^-(\omega)$ which is θ invariant a.s. On the other hand the right hand side has the same distribution like

$$-\frac{1}{n}\log\delta_2(\Phi(n,\omega))$$

which converges to $-\lambda_2(\omega)$ which is also θ invariant a.s. We can conclude that $\lambda_2(\omega) = -\lambda_1^-(\omega)$ a.s. Similar we obtain $\lambda_1(\omega) = -\lambda_2^-(\omega)$ such that one of the assumptions (20) is satisfied. The other assumptions are given by the following arguments. Let now $\lambda(x,\omega)$ be the Lyapunov exponent for $\Phi(n,\omega)$. Then similar to the proof of Lemma 37 we have

$$\lambda(x,\omega) = \lambda(\Phi(1,\omega,\theta_1\omega)) \le \lambda_i(\omega) = \lambda_i(\theta_1\omega)$$

which means

$$\Phi(1,\omega)x \in V_i(\theta_1\omega) \quad \text{or } \Phi(1,\omega)V_i(\omega) \subset V_i(\theta_1\omega)$$

for $i = 1, 2$. We can replace the later inclusion by an equality because $\Phi(1,\omega)$ is non singular. In particular, we have

$$V_2(\omega) \cap V_2^-(\omega) = \{0\}.$$

which follows because these spaces are one dimensional and $\lambda_1(\omega) \neq \lambda_2(\omega)$. Ergodicity and and invariance of the λ_i that these values are independent of ω. We are able to prove that the spaces V_i are related to these values.

REMARK 41. *(i) We note that for the general case the proof is more complicated. The the Lyapunov exponents and the space V_i can be derived from*

$$\Psi(\omega) = \lim_{n\to\infty}(\Phi(n,\omega)^T\Phi(n,\omega))^{1/2n}.$$

(ii) There is a version of the Oseledets theorem for linear RDS in Hilbert spaces but also in Banach spaces, see [**MZZ08**], [**LL10**].

An example generating an linear RDS Φ on $\mathbb{T} = \mathbb{R}$ is the linear random differential equation

$$\frac{du}{dt} = A(\theta_t\omega)u, \quad u(0) = x$$

where $\|A\| \in L_1(\Omega)$. Applications showing in the direction to construct invariant manifolds then can be found in [**CDLS10**].

REMARK 42. *The integrability condition for the Oseledets theorem for a linear random differential equation* (**??**) *is satisfied if the generator $A(\omega)$ is in $L_1(\Omega)$.*

As an example we now consider the two dimensional random linear differential equation (**??**)
where

$$A(\omega) = \begin{pmatrix} a_1(\omega) - L(\omega) & -L(\omega) \\ L(\omega) & a_2(\omega) + L(\omega) \end{pmatrix}, \quad L(\omega) > 0.$$

The dynamics of this equation can be described by the Oseledets theorem 40. The system is related to some problem of random invariant manifolds.

LEMMA 43. *Suppose that $A \in L_1(\Omega, \mathcal{F}, \mathbb{P})$. In addition we suppose that*

$$\mathbb{E}(a_1 + a_2) < 0. \tag{22}$$

and that the gap condition

$$a_1(\omega) - a_2(\omega) - 4L(\omega) > 0 \tag{23}$$

holds θ-a.s. Then the dynamical system ψ generated by (6) has two different Oseledets spaces θ-a.s. The Oseledets space E_1 (related to the biggest Lyapunov exponent) has an angle between zero and $\pi/4$. The second Lyapunov exponent λ_2 is negative.

PROOF. We consider the dynamics of (**??**) projected onto the unit sphere. The angle α of the solution of this two dimensional system satisfies the following differential equation

$$\begin{aligned} \alpha_t' &= (A(\theta_t\omega)(\cos\alpha_t, \sin\alpha_t)^T, (-\sin\alpha_t, \cos\alpha_t)^T) \\ &= \frac{1}{2}(a_2(\theta_t\omega) - a_1(\theta_t\omega) + 2L(\theta_t\omega))\sin(2\alpha_t) + L(\theta_t\omega) \end{aligned} \tag{24}$$

see Arnold [**Arn98**] Page 278. For $\alpha = 0$ we have $\alpha_0' > 0$ and for $\alpha = \frac{\pi}{4}$ we have by the gap condition (23) $\alpha_{\pi/4}' < 0$. Therefore $[0, \pi/4]$ is forward invariant for (24). This forward invariance garantees the existence of a random attractor on the sector $[0, \pi/4]$ which is an interval. By the discussion below for negative time this interval is degenerated, a single point attractor. Hence there is a stationary point of the dynamical system generated by (24) on $[0, \pi)$. This stationary angle between 0 and $\pi/4$ represents an Oseledets space. If we reverse the time in (24) we find another stationary point in $[\pi/4, \pi/2)$ representing the second Oseledets space. By (23) $\pi/4$ does not represents a stationary point such that we have found two different Oseledets spaces. These Oseledets spaces are represented by stationary points (or stationary angles) $\hat{\alpha}$, $\check{\alpha}$.
By the attractor properties of these random stationary points we find that $\hat{\alpha}$ is related to the the Lyapunov exponent λ_1 which is the larger Lyapunov exponent for $t \to \infty$. According to Birkhoff's and Liouville's theorem, see Arnold [**Arn98**] Page 60, we have that

$$\lambda_1 + \lambda_2 = \lim_{t\to\infty} \frac{1}{t} \log|\det\Phi(t,\omega)| = \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(\theta_\tau\omega)d\tau = \mathbb{E}(a_1 + a_2) < 0$$

such that λ_2 is negative. It follows that α_t tends to $\check{\alpha}(\theta_t\omega)$ for $t \to -\infty$ and hence α_t tends to $\hat{\alpha}(\theta_t\omega)$ for $t \to \infty$. ☐

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